CHAPTER 4

THE GAP IN THE TRANSITION FROM ELEMENTARY TO ADVANCED THINKING

4.1 Introduction

There is a significant difference between mathematical thinking and general thinking (Alcock & Simpson as quoted by Engelbrecht, 2010:144). The process of "doing problems" and "proving statements" are very different and require a change in thinking styles. According to Engelbrecht (2010:144), Alcock and Simpson indicate in their study that mathematical thinking (including deductive reasoning) is not something that comes naturally, but is a skill that is a prerogative for advanced mathematics. For students to acquire mathematical reasoning, they have to be explicitly taught those skills. From his point of view students are used to general thinking where facts are accumulated to prove a particular statement. In contrast with the general way of thinking, mathematical thinking is based on an axiomatic system in which deductive reasoning (see par 3.5.3.1) is the only route. Concepts are defined and theorems are proved using the rules of logic. Although empirical induction is used to arrive at conjectures and the role of discovery, explanation and analytical methods are important, actual proof depends highly on logical deduction. In advanced mathematical thinking (AMT) students must be able to work with definitions and theorems and apply deductive reasoning, which only a few students can do naturally (Engelbrecht, 2010:143).

This chapter investigates the gap between secondary and tertiary mathematics in terms of the journey from elementary to advanced thinking. When studying cognitive development, specifically in a mathematics domain, the names of Piaget, Van Hiele and Tall are of importance. One can distinguish between two kinds of theories of cognitive development namely global and local theories (Pegg & Tall, 2005:468). *Global theories* of conceptual growth address the growth of an individual over the long-term. They start with the initial interaction of a young child with the world through to the development of new ways of thinking as the individual matures. Underlying these global perspectives is the gradual biological development of the individual. The theories of Van Hiele and Piaget can be classified as *global* frameworks of long-term growth. *Local theories* relate to a specific conceptual area in which a learner/student attempts to make sense of the information available and to make connections using structures available to them e.g. differentiation. The theory of Tall can be classified as a *local* framework of conceptual growth.

The views of Piaget, Van Hiele and Tall on conceptual development in particular will be further discussed.

4.2 Jean Piaget's cognitive development theory

Jean Piaget's cognitive theory discusses how an individual progresses through the learning process in stages. These stages consist of a period of months or years when certain development takes place. The order of the stages is constant and sequential and the sequence never differs. Piaget believed that children develop steadily and gradually through the various stages and that the experiences in one stage form the foundations for the next stage (Ojose, 2008:26).

Piaget identified four stages of cognitive development:

Sensorimotor stage: This stage is generally between birth and 2 years old. In this stage children begin to link numbers to objects and start counting on their fingers. They understand that one finger matches with the number one and can count concrete objects using this concept (Ojose, 2008:27).

Pre-operational stage: Between 2 and 7 years of age children are able to do one-step logical problems (Blake & Pope, 2008:60) and this stage marks the beginning of solving mathematically based problems like addition and subtraction. These children are restricted to one dimensional thinking and will be influenced by the visual representation of things.

Concrete operational stage: In this stage (7 to 11 years) a child is able to think logically and can start classifying based on several features and characteristics. They can consider two or three dimensions simultaneously instead of successively. Seriation (the ability to order objects according to its length, height, weight, volume) and classification (the ability to group objects on the basis of common characteristics) are the two logical operations that develop during this stage. Both these operations are essential for understanding number concepts (Ojose, 2008:27).

Formal operational stage: Children enter this stage roughly between 11 and 16 years and it continues throughout adulthood. At this stage a child develops abstract thought patterns. The reasoning skills in this stage refer to generalizing and evaluating of logical arguments (Ojose, 2008:28). They are capable of forming hypotheses and to deduce possible consequences, as well as to think inductively and deductively. At this stage a person can argue by implication e.g. if x then y (Sinclair, 1974: 76). People who have reached this intellectual stage can view definitions, rules and laws in a proper objective context (Bell, 1991:100). The child will therefore be able to understand abstract concepts, which leads to complicated mathematical thinking.

Piaget explains intellectual development as a process of assimilation and accommodation of information into the mental structure. *Assimilation* is the process through which new information and experiences can be placed in pre-existing cognitive structures and *accommodation* is the process of changing cognitive structures in order to accept something from the environment (Blake & Pope, 2008:61). The mind does not only receive new information and experiences, but it restructures its old information to accommodate the new. This implies that learning is not only the addition of new information to old information, but requires old information to be modified to accommodate the assimilation of new information.

According to Piaget's theory there are five processes that enable the transition from one stage to another (Bell, 1991:100-101). *Maturation*: The physiological growth of the brain and the nervous system determines the possibilities of the different stages. If a stimulus is presented prior to a certain level of competence, the stimulus will have no value at all. It means that a person cannot undertake certain tasks until they are psychologically mature enough to do so. *Physical experience* consists of the interaction of a person with objects in the environment and the process of drawing some information about the object by abstraction. *Logical mathematical experience* is the experience of the actions of the subject and not experience of the objects themselves. It is the mental actions performed by individuals as their mental schemas are restructured according to their experiences. *Social transmission* is the interaction and cooperation of a person with other people. Piaget believed that formal operations would not develop in the mind without exchange and coordination of viewpoints among people. In Piaget's view all children try to strike a balance between assimilation and accommodation, which is achieved through a mechanism he called *equilibration*. As children progress through the stages of cognitive development, it is important to maintain a balance between applying previous knowledge (assimilation) and changing behaviour to account for new knowledge (accommodation). Equilibration helps explain how children are able to move from one stage of thought into the next. It can be seen as the force that moves development along. An unpleasant state of disequilibrium happens when new information cannot be fitted into existing schemas (assimilation). Equilibration is the force that drives the learning process as people do not like to be frustrated and will seek to restore balance by mastering the new challenge (accommodation). Once the new information is acquired the process of assimilation with the new schema will continue until the next time the person needs to make an adjustment to it.

Piaget (1997:23) indicates that "*the child can receive valuable information via language or via education directed by an adult only if he is in a state where he can understand this information. That is, to receive the information he must have a structure which enables him to assimilate this information*". Piaget (1974:120–122) says that a feeling of necessity where the child will say "but this is obvious", indicates the closure of a structure of a certain stage. He believed that children develop through the different stages and can only go on to the next stage if they pass the previous stage successfully. This implies that older children and adults who have not passed through a certain stage of development process information in ways that are characteristic of younger children in that developmental stage.

4.3 The Van Hiele levels of the development of geometric thought

Pierre M. van Hiele and his wife Dina van Hiele-Geldof made an important contribution to the teaching of geometry with their Theory of Cognitive levels in Geometry. Even though Van Hiele stated that parts of his work can be found in the theories of Piaget (Van Hiele, 1986:5), he disagreed with certain aspects of Piaget's theory. Van Hiele believed that the theory of Piaget was one of development and not one of learning. Piaget did not accommodate the problem of how to stimulate children to go from one level to another, while Van Hiele did. He was also convinced that accommodation of learning in geometry requires more than two levels of understanding, while Piaget distinguishes only two levels in this phase (pre-operational and concrete operational). Van Hiele (1986:4-5) is convinced that the development of insight is the main purpose of learning. Piaget believed that a person needs to achieve a certain level (the formal operational stage) to be able to reason formally and understand and construct proofs. Van Hiele believed that the development of mathematical reasoning can be accelerated by instruction.

Van Hiele focussed in his explanation of his theory of cognitive levels on geometry and postulated five levels of geometric reasoning. These levels are labelled as visual, analysis, informal deduction, formal deduction and rigour (Crowley, 1987:9). He described characteristics of the thinking processes necessary for geometric understanding. These levels are not age dependent like the stages Piaget described, but are related to the experiences students have had. They are sequential - students must pass through the levels as their understanding increases. If a teacher tries to teach a student at one level when a student has not passed through the previous level, the student will not understand the teacher and this will result in rote learning. "*The transition from one level to a next level is not a natural process; it takes place under the influence of a teaching-learning program"* (Van Hiele, 1986 as indicated by Teppo, 1991:210). It means that instruction, rather than maturation, is the most significant factor contributing to this development (Crowley, 1987:23). Thus Van Hiele traced cognitive development through a succession of increasingly sophisticated levels (Tall, 2004b:282).

4.3.1 The Van Hiele levels of reasoning

Van de Walle (2007:409-412) summarizes the levels of reasoning as follows:

Level 0: *Visualization*. Students at this level recognize and name figures based on the global characteristics of the figure. Students have a picture of the shape in mind and the appearance of the shape defines it for the student.

Level 1: *Analysis*. The student recognizes properties of a figure and forms classes of figures, e.g. rectangles. Students are able to think about what makes a rectangle a rectangle (four sides, opposites sides parallel, opposite sides same length, four right angles, etc.).

Level 2: *Informal deduction*. Students are able to develop relationships between and among properties. They can order and relate properties, for example a square is also a rectangle. Students at this level will be able to follow and appreciate an informal deductive argument about shapes and their properties.

Level 3: *Deduction*. A structure with axioms, definitions, theorems, corollaries and postulates begins to develop and can be appreciated. At this stage students see the need for a system of logic from which other truths can be derived. The type of reasoning that characterizes this level is the same as is typically required in secondary school geometry courses.

Level 4: *Rigour.* The student understands formal aspects of deductions. Axiomatic systems can be constructed, for example an axiomatic system for spherical geometry.

Van Hiele (1999:310) argues that the teaching of geometry in secondary school is most of the times presented in an axiomatic fashion that assumes that students think on a formal deductive level. However, this is usually not the case and the students lack prerequisite understandings about geometry.

Usiskin (1982:4-5) describes the properties of the levels identified by Van Hiele as follows:

Property 1: *Fixed sequence*: It is inherent in the Van Hiele theory that a person must go through the different levels in order. A student cannot be at level n without having gone through level n-1.

Property 2: *Adjacency*: What was intrinsic in the preceding level becomes extrinsic in the current level.

Property 3: *Distinction*: Each level has its own linguistic symbols and its own network of relationships connecting those symbols.

Property 4: *Separation*: Two persons who reason at different levels cannot understand each other.

Usiskin explains these properties using the following example: A students' remark to his/her teacher: "I can follow a proof when you do it in class, but I can't do it at home". The student may be at level 3, while the teacher is operating at level 4. Property 4 indicates that the student cannot understand the teacher and property 3 explains why there is no understanding. The teacher is using objects (propositions, in the case of proof) and a network of relationships (proof itself) that the student does not yet understand and use in this way. If the student is at level 3, then the student's network consists of simple ordering of propositions and property 2 indicates that these orderings, intrinsic at level 3, become extrinsic at level 4.

4.3.2 Stages of instruction to reach a next level

In the learning process, progress form one level to another involves five stages, namely (Van Hiele, 1986:53-54):

- Information: The student gets acquainted with the working domain (e.g. examines examples and non-examples).
- Guided orientation: The student does tasks involving different relations of the network that has to be formed (e.g. folding, measuring).
- Explicitation: The student becomes conscious of the relations; he or she tries to express them in words and learn technical language that accompanies the subject matter (e.g. expresses ideas about the properties of figures).
- Free orientation: Students learn to find their own way of doing in the network of relations (e.g. knowing properties of one kind of shape, they can investigate properties for a new shape).

- Integration: The student builds an overview of all he or she has learned of the subject with a newly formed network of relations now available.

A critical aspect of Van Hiele's theory is that a person can only progress from one lower level to a next if properly "facilitated" through an appropriate learning period comprising the above stages. If not, that person will remain on the lower level irrespective of ageing. Also, a person can progress in some aspects (e.g. algebraic thinking) to a next level, while remaining on the lower level in other aspects (e.g. proof).

4.3.3 Reduction of the five levels

Van Hiele (1986:53) suggests that a simplified classification of the levels is possible. He comments as follows: "The above classification is suitable to a structure of mathematics and perhaps mathematicians will be able to work with it". The first three levels of this model are especially suitable to apply in the school environment. According to Teppo (1991:210) the reduced levels are the visual level, descriptive level and the theoretical level. The descriptive level includes the analysis and the informal deduction levels. On this level students distinguish shapes on the basis of their properties. The theoretical level includes the levels of deduction and rigour. At this level students are able to devise a formal geometric proof. This level has an abstract character where reasoning about logical relations takes place.

4.4 Tall's three worlds of mathematical thinking

The teaching of mathematics to university students involves the introduction of different ways of thinking applicable to the total field of mathematics. Tall and Mejia-Ramos (2006) state that cognitive development builds on experiences that were *met-before*. They define a *met-before* as part of an individual's concept image in the form of a mental construct that is based on experiences they have met before. This occurs through neuronal connections that strengthen successful links and suppress others. Tall (2008:6) defines a met-before as a current mental facility based on specific prior experiences of the individual. Met-befores can affect the way individuals interpret new mathematics, sometimes it is an advantage, but other times it can cause internal confusion that impedes learning. The brain changes its ability to think over time and reorganizes information to create new structures to cope with new situations. Experts need to reflect on how different students' met-befores affect their ways of learning.

Tall (2007) sees the cognitive development of an individual in terms of three distinct but intertwined worlds, namely the conceptual-embodied world, the proceptual-symbolic world and the axiomatic-formal world. The word "world" is used to represent the development of distinct ways of thinking that becomes more sophisticated over time. The worlds describe different ways of thinking that individuals develop as new conceptions are compressed into more thinkable concepts (Tall & Mejia-Ramos, 2006).

The *conceptual-embodied world* or "embodied world" is where we think about the things around us in the physical world. It does not only include mental perceptions of real-world objects, but also our internal conceptions that involve our own mental world of meaning. It is based on the perception of and reflection on properties and objects, patterns, verbal descriptions and definitions that formulate relationships and deductions (Tall, 2004a:30). Conceptual embodiment grows from perception of objects, through description, construction and definition leading to deduction (Tall, 2008:7).

The proceptual-symbolic world or "symbolic world" develops out of the embodied world through actions such as counting, sharing and measuring. It begins with actions such as pointing and counting and involves practising actions until they can be performed accurately with little conscious effort. It develops beyond the learning of procedures to carry out a process to the concept created by that process (Tall, 2010:22).

The *axiomatic-formal world* or "formal world" is based on properties expressed in terms of formal definitions that are used as axioms (Tall, 2004a:30). Tall claims that it turns the previous experiences on its head, not working with familiar objects, but with axioms that are formulated to define structures in terms of specified properties. Other properties are then deduced by formal proof to build a sequence of theorems. The formal world arises from a combination of embodied conception and symbolic manipulation.

Mathematical thinking switches from the embodied to the symbolic world. A curriculum that focuses on symbolism and not on related embodiments may limit the vision of the learner who may learn to perform a procedure, but fails to compress it into thinkable concepts to be used flexibly for more sophisticated thinking. The divergence in success and failure in mathematics can be related to the development of the notion of procept (Gray *et al*., 1999:120). Gray and Tall (1994:121-122) use the word procept to refer to a process and a concept represented by the same symbol. A symbol such as $\int sinxdx$ represents a process, the integral of $sinx$ to be carried out and the concept of integration produced by that process. They suggest that interpretations of mathematical symbolism as process or procept leads to a proceptual division between less successful and more successful students. On the one hand there is a strong association with procedures and on the other a style more in tune with the flexible notion of procept. Those using the latter have an advantage as they derive mathematical flexibility from the cognitive links relating process and concept. The ability to use a specific procedure for a specific purpose allows the individual to do mathematics in a limited way. Some may develop a greater sophistication by being able to use alternative procedures for the same process and to select more efficient procedures for a specific task to be carried out accurately. It might therefore be possible for individuals at different levels of sophistication to answer a test successfully. However, this may not be an indication of success at a later level because the procept in its distilled form is more ready for building into more sophisticated theories than step-by-step procedures (fig. 4.1). Those who operate successfully at the procedural level are faced with much greater complexity than their proceptual friends when the next level of difficulty is encountered (Gray *et al*., 1999:121-122).

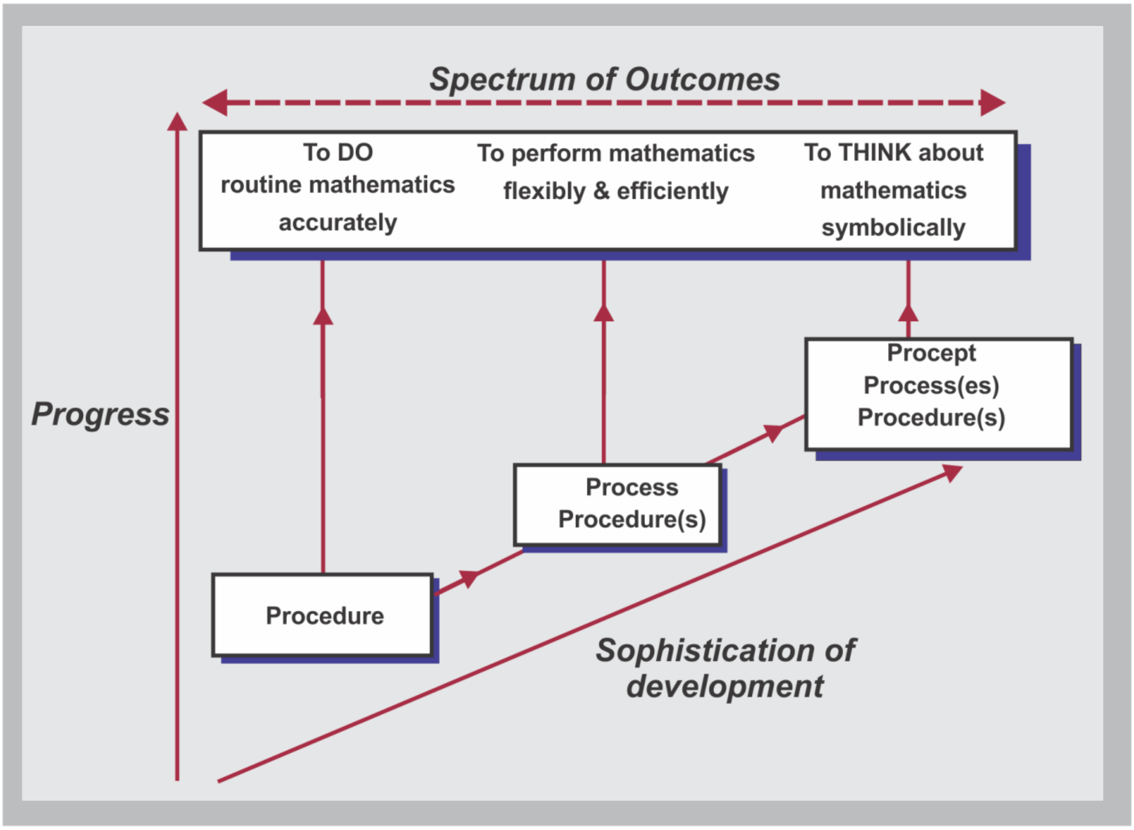

Fig 4.1: **A spectrum of performance in using mathematical procedures, processes and procepts (Gray *et al.*, 1999).**

If tertiary courses try to build thinking in the formal world with students who are symbolic thinkers, then difficulties will arise.

4.5 Integrating Piaget, Van Hiele and the three worlds of mathematics of Tall

Mathematical understanding develops through different levels according to Van Hiele (par. 4.3) and Piaget (par. 4.2). They both suggest that students should pass through lower levels of thought before they are ready to move on to the next level. For Piaget there is a maturation process that takes place and for Van Hiele students progress through given levels as a result of experience. Van Hiele (1999:310) supports Piaget's point of view that "*giving no education is better than giving it at the wrong time*", underlining the fact that students cannot bypass levels or stages and achieve understanding. Battista and Clements (1995) argue that premature dealing with formal proof can only lead students to attempt memorisation instead of understanding how to execute proofs. This confirms Piaget's and Van Hiele's reasoning that students have to master a level before moving on to the next level. Piaget and Van Hiele both emphasize the role of active engagement in constructing knowledge. Van Hiele emphasizes that successful students do not learn facts, names or rules, but networks of relationships that link concepts and processes (Clements & Battista, 1992:436).

The communality among the theories of Piaget, Van Hiele and Tall is that they all include some form of hierarchical structure (table 4.1). Van Hiele, as well as Tall, argues that the first level of mathematical understanding starts with visualisation or "perceptual representations of concepts". A person conceptually embodies a geometric figure such as a triangle as consisting of three straight line-segments; the person imagines such a figure in his or her mind and allows this specific triangle as a prototype to represent the class of triangles (Tall, 2008:7). Or the person *sees* an image of a specific graph as representing a specific function. Tall agrees with Van Hiele that the conceptual embodiment gets more sophisticated as a student matures with specific instruction.

The second level (Van Hiele) or world (Tall) refers to changing what you see in your mind to symbols or words. According to Tall (2008:8) symbols such as 3 + 2 represents both a process to carry out (addition), as well as a concept of the "sum of" produced by the process. On the descriptive level Van Hiele found that students do not only think about properties of a figure, but are able to notice relationships within and between figures. They use verbal skills to communicate and formulate meaningful definitions.

The third level or world refers to formalism where axioms, definitions and theorems are prominent. From Tall's point of view the transition to the formal world builds on the experiences of embodiment and symbolism to formulate formal definitions and to prove theorems using mathematical proofs (Tall, 2008:10). According to him the written formal proof is the final stage of mathematical thinking. The third level of Van Hiele's model refers to the use of reasoning to prove geometrical relationships. On this level students are able to devise a formal proof and are able to understand the processes employed (Teppo, 1991:211).

Table 4.1 Local and global stages of cognitive development

LOCAL FRAMEWORK	GLOBAL FRAMEWORK	
Tall	**Piaget**	**Van Hiele**
Embodied world – thinking about things in the physical world, as well as in the mental world. Symbolic world – the world of symbols that is necessary for calculation and manipulations Formal world – based properties expressed in terms of formal definitions that are used as axioms.	Sensori-motor – infancy. Preoperational – early childhood through preschool. Child is able to use symbolic thought. Concrete operational – childhood to adolescence. Logical reasoning is based on concrete objects. Formal operational – early adulthood. Can reason about numerous possible solutions and are able to think abstractly.	Visual level – Recognise geometric objects globally. Descriptive level – Recognize an object by its geometric properties. Theoretical level – Think in terms of abstract mathematical systems.

Tall *et al*. (2001:102-103) indicate that the journey from elementary mathematics through to university mathematics has a range of discontinuities. They suggest that mathematics cannot be structured as a simple curriculum built on old foundations in established ways, but requires constant rethinking to cope in new contexts. They also argue that there is not a single way of teaching mathematics if taking into account the different ways students' thinking develops and how they learn. Therefore the human interface between teaching and learning is a constant source of renewal.

4.6 The shift from elementary to advanced mathematical thinking

Elementary mathematical thinking is characterized (Gray *et al*., 1999:116) as the use of symbols as concepts and processes to calculate and manipulate. It begins with actions on objects in the physical world and requires the focus of attention to shift from the action of counting to the manipulation of number symbols. From here there is a movement towards generalisations and more sophisticated symbol manipulation in algebra and calculus. Another kind of cognitive development in elementary mathematics is the Van Hiele development of geometric objects and their properties from physical perceptions and geometric objects. Objects are described and the description is constructed from experience of the object (Tall, 1995). The meaning of the objects and symbols comes from the experience of playing with them and finding out their properties (Tall *et al*., 2001).

In advanced mathematics the emphasis is on abstraction and proof. Advanced mathematical thinking is characterized (Tall, 1992) as involving both precise mathematical definitions and logical deductions of theorems based upon these. Many mathematics educators use the phrase “advanced mathematical thinking” as something that would describe certain kinds of students thinking at the professional level of mathematics. It seems as if it first occurs when a student begins to deal with abstract concepts and deductive proof. Edwards *et al*. (2005:18) defines it as deductive and rigorous reasoning about mathematical ideas that are not entirely accessible through the five senses. Tall (1992) links it to formal mathematics. He characterizes advanced mathematical thinking as consisting of precise mathematical definition and logical deductions of theorems based on them.

Because advanced mathematical concepts dealt with at university are the result of several abstraction sequences, the network of relationships among concepts can be very complex and requires conceptual understanding that reflects a “deeper” approach. It seems as if the inability to link the conceptual with the procedural is the root of students’ difficulties with higher level mathematics. Jooganah and Williams (2010:114) are of the opinion that practices that students are expected to engage in at university, which emphasize proof and the need for rigour, may cause cognitive conflict between old ways of doing and new demands made by university practices.

The rigorous and abstract nature of mathematics gaining dominance over the heuristic and concrete approaches in secondary school, as well as the formidable formalism play a crucial role in the inadequate transformation from secondary to tertiary mathematics (Luk, 2005:162). Educators need to understand the mental processes needed by students to understand the abstract concepts, to master the formal language, to follow rigorous reasoning and to acquire mathematical maturity. Mathematical thinking is psychological and this means that the understanding of the psychology of mathematical thought would be a necessary first step in bridging the gap for students. Mathematicians may take the mathematical way of thinking for granted, unaware of the fact that they may be talking in a foreign language to their students.

According to Edwards *et al*. (2005:18) there is not a point when elementary mathematical thinking ends and advanced mathematical thinking begins. We can consider the first level beyond elementary school mathematics to be a preliminary stage of advanced mathematical thinking in which elementary ideas are stretched to their limits (Tall, 1995:174). In elementary mathematical thinking arithmetic begins by counting actual objects and properties such as associativity, commutativity and distributivity are directly experienced by the individual. Students have encountered objects that possess properties and symbols that can be manipulated. The meaning of objects and symbols comes from the experience of playing with them and finding out their properties. In formal mathematics students deal with definitions in words and symbols that build new mathematical entities through deduction, building up their properties through a sequence of theorems and proofs. Students have difficulty to cope with this new formal theory where everything must be deduced from definitions (Tall *et al*., 2001:97).

The procedural experiences and the general ways of thinking in elementary mathematical thinking support ways of understanding and thinking in advanced mathematical thinking and should be fostered during the study of elementary mathematics (Harel & Sowder, 2005:46). The use of procedures that are specific sequences of steps carried out one step at a time allows someone to do a specific computation or manipulation. Those who are procedurally oriented are limited to a particular procedure with focus on the steps itself and have a greater burden to face in learning new mathematics (Tall *et al*., 2001). Mathematics does not consist of isolated rules, but connected ideas. Having one or more alternatives available allows greater flexibility and efficiency to choose the most suitable route for a given purpose. Elementary mathematics is rich with opportunities for students to develop a repertoire of reasoning and to build connections and should not be ignored until students take advanced mathematics courses (Harel & Sowder, 2005:41).

There is a cognitive discontinuity from the elementary mathematical thinking of calculation and manipulation to the advanced mathematical thinking of defining and proving. Therefore a move from elementary to advanced mathematical thinking requires a cognitive restructuring of a student's thinking processes. The transition to advanced mathematical thinking makes a complete shift from the existence of perceived objects and symbols representing actions on the objects, to new theories based on specified properties of formally defined mathematical structures.

Students can have different strategies in dealing with advanced mathematical thinking of deduction and formal definitions (Tall *et al*., 2001). Some give meaning to the definition by manipulating their mental imagery and by constructing examples and non-examples to build up meaning for the definition and building a range of possibilities that might be deduced from the definitions. Others may extract meaning from the definition by formal deduction in proving theorems with little or no intervening of imagery. This strategy can build up a formal theory that is more confined in itself and less linked to other aspects of the student's cognitive structure.

In a study done by Stewart and Thomas (2009:960) it appears that students often lack the embodied aspects of concepts, which hinders them from moving to the formal world of mathematical thinking. The cognitive development through the three worlds of mathematics should start from the embodied world through to the symbolic world to finally arrive at the formal world. However, in the real world this does not always happen. A visual embodied approach should enrich students' understanding and give them a better foundation for symbolic and formal thinking. Students experience a significant amount of conflict in their first year because mathematics at university level shifts from visual and symbolic representations towards a formal framework of axiomatic systems and proof (Tall,1995). By investigating students' learning preferences compared with their beliefs about the learning of mathematics, one should get more information on the gap in the transition from elementary to advanced mathematical thinking.

4.7 Conclusion

At the end of this chapter one can conclude that a teacher at secondary school or a lecturer at university of mathematics should take into consideration that students and learners in secondary schools go through different stages of cognitive development on their journey to learning mathematics with understanding. The next chapter provides an outline and description of the empirical study, as well as the results of the different parts of the study.