CHAPTER 3

THE GAP BETWEEN SECONDARY AND TERTIARY LEVEL CONCERNING BELIEFS ON THE STRUCTURE OF MATHEMATICS

3.1 Introduction

The shift from a modern to a post-modern paradigm, as well as a change in the whole education system, cause first year students at tertiary institutions in South Africa to exhibit a different view of the nature of mathematics and its learning than their lecturers, as well as students from the previous school curriculum. During the last century a shift in focus has occurred in mathematics education from just knowing mathematics to doing mathematics. Results from a South African study on the beliefs of the nature of mathematics showed that the participating teachers in that study tended to be innovative, which correlates with the innovative views of teaching and learning practices in the constructivist paradigm (Webb & Webb, 2004:13). However, despite the teachers' constructivist beliefs, their classroom practices tend to be very traditional. Another study on the content knowledge of mathematics teachers in a specific professional development program revealed that their mathematical knowledge is overshadowed by unconnected procedural knowledge, which hinders them to be good teachers of mathematics (Plotz, 2007:237). Even though the NCS (DoE, 2003) proposed that learners emerging from the FET band should demonstrate the ability to think logically and analytically and be able to transfer skills from familiar to unfamiliar situations, first year students struggle to adapt to tertiary mathematics where they need these skills.

This chapter investigates how thoughts on the structure of mathematical knowledge have changed to come to an understanding of how this change will influence the transition from secondary to tertiary level.

3.2 Views on the structure of mathematics

Belief systems are based on one's mathematical worldview, the perspective with which one approaches mathematics and mathematical tasks. How teaching takes place is strongly based on an educator's understanding of the nature of mathematics. Within the frame of the static-formalist view mathematics is seen as a fixed and logical network of inter-related facts, rules and algorithms and learning is perceived as the internalizing of neatly dissembled blocks of knowledge that learners or students should practice and memorize as facts, skills and procedures. In order to facilitate such learning, educators should unfold this structure of knowledge and deliver it in neat blocks through explanation and demonstration. The fundamentals of the new post-modern paradigm lead to a radically different view of mathematics as a dynamic structure that is part of human experience and interaction. Mathematics is not a "finished" product with its origin outside the individual, but remains "in the making" in the individual's mind. For learners and students to really understand mathematics they need opportunities to communicate meaningfully about mathematics and form connections between important mathematical ideas and concepts. They should be engaged in critical thinking about ways in which mathematics may be used in everyday life (Ellis & Berry, 2005:11-12).

According to Hourigan and O'Donoghue (2007:464) the nature of mathematics teaching in secondary schools directly influences the ability of learners to successfully make the transition to tertiary mathematics courses. Much of the teaching done in South African schools are still traditional in nature with a focus on procedures and not on understanding. Changing the beliefs of educators is a challenge (Cheng *et al.*, 2009:319). Curriculum trends towards reform require educators to change their beliefs and practices in the classroom, even though most of the time they have no intention to change. This demands a process of "unlearning" of what has long been held as correct and then "learning again" what is new, which can bring a sense of loss, anxiety and struggle (Harley *et al.*, 2000:300; Mousley, in Handal & Herrington, 2003:62).

Plotz (2007) developed a model that can be used to restructure the mathematical content knowledge of teachers in order to enable them to teach mathematics for understanding. This model assists teachers in changing from a one-dimensional structure, consisting mainly of procedural or algorithmic knowledge, to a multidimensional structure consisting of various integrated components. According to the model the cognitive processes of representation, connection, reasoning and transfer to different problem contexts is given the same weight as procedural and conceptual knowledge to emphasize that these cognitive processes become knowledge *per se* (Plotz *et al*., 2012:77). The components of Plotz's model can also act as a vehicle to investigate the structure of mathematics in terms of a constructivist paradigm.

The next three paragraphs address these components, namely what it means to understand mathematics, what procedural and conceptual knowledge entails, and the different processes through which mathematical understanding develops.

3.3 Mathematical understanding

Understanding may be defined differently by various people. According to Skemp (1978:9) some may consider understanding as the possession of a rule and the ability to use it. He calls this kind of understanding "*instrumental understanding*", which implies the application of "*rules without reasons*". If students learn to divide by a fraction they learn that one only has to "*turn it upside down and multiply*" without knowing why this has to be done. Skemp argues that for instrumental understanding one has to learn many rules instead of a few principles of more general application. Skemp realised that there are many teachers and learners that think that to understand mathematics one must only get the right answer to a problem. Instrumental understanding leads to rote learning. Rote learning produces knowledge that has no relationships with other knowledge and is tied to the context in which it is learned. The knowledge that results from rote learning is not linked with other knowledge, and therefore does not generalize to other situations and can only be applied to contexts that look like the original context (Hiebert & Lefevre, 1986:8).

Relational understanding is defined by Skemp (1978:9) as knowing what to do and why. He characterizes relational understanding of mathematics as the possession of conceptual structures that enable a person to construct plans for performing a given task. In learning relational mathematics, the learner has to become independent from the particular end to be reached. It includes insight into the why of mathematics. From Van de Walle's (2007:25) point of view understanding of mathematics is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding will be.

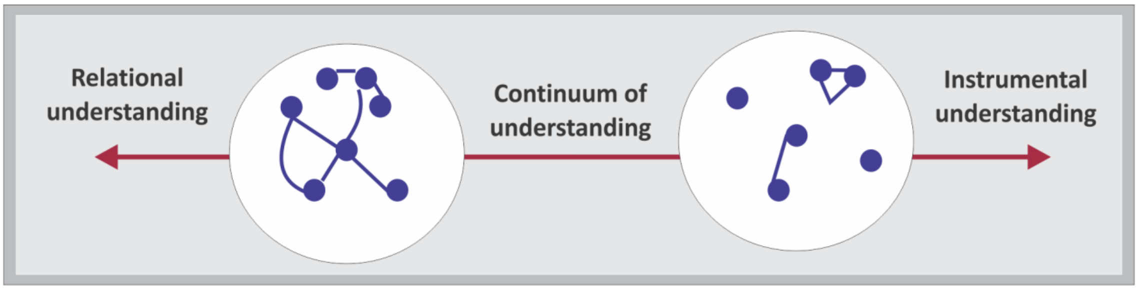


Fig. 3.1 A representation of relational and instrumental understanding (Van de Walle, 2007:25)

The kind of learning that leads to instrumental understanding of mathematics consists of an increasing number of fixed plans, by which learners can find their way from the beginning to the end (the answer) (Skemp, 1978:14). The plans tell them what to do at each choice point, what has to be done next is determined by the local situation – e.g. when you have cleared the brackets, collect like terms. A learner is dependent on external guidance for each new action to get to the answer. In contrast with this, learning mathematics relationally consists of building up a conceptual structure from which the possessor can produce an unlimited number of plans for getting from the starting point to any finishing point.

Advantages of instrumental understanding are (Skemp, 1978:12-13):

- *It is usually easier to understand*. Some topics are difficult to understand relationally instead of just learning the rule. If one only wants a page with correct answers, the instrumental way of understanding is quicker and easy. It would be easier to learn the rule for dividing a fraction, turn it upside down and multiply, as to understand the concept of why one has to do this.

- *The rewards are immediate*. A page with correct answers may enhance a learners' self-confidence as success helps to restore a persons' self-confidence.
- *Less knowledge is involved*; one can often get to the correct answer quicker.

Advantages of relational understanding are (Skemp, 1978:12-13):

- *It is more adaptable to new tasks*. Knowing not only what method will work for a problem but why it will work, makes it possible to adapt to new situations and new problems.
- *It is easier to remember*. There are not many rules to learn for every situation.
- *Relational knowledge can be effective as a goal in itself*.
- *Relational schemas are organic in quality*. If someone gets satisfaction from relational understanding, they might always try to understand relationally and seek for new material to explore new areas.

The fundamental question in teaching is: What are the desired learning outcomes? Do you want to teach for instrumental understanding or for relational understanding? If learners are taught instrumentally, then the "traditional" syllabus will probably benefit them more, whereas if learners are taught relationally, then it would best fit in with the dynamic, problem-solving view of mathematics.

3.4 Procedural and conceptual knowledge

Procedural knowledge of mathematics is knowledge of the rules and procedures that one uses to carry out routine mathematical tasks, as well as the symbolism that is used to represent mathematics (Van de Walle, 2007:28).

Procedural fluency is described by the New York State Education Department (NYSED, 2009) as "*the skill in carrying out procedures flexibly, accurately, efficiently, and appropriately. It includes, but is not limited to, algorithms (the step-by-step routines needed to perform arithmetic operations)"*. Hiebert and Lefevre (1986:6) describe procedural knowledge as "*made up of two distinct parts. One part is composed of the formal language or symbol representation system ….. [and] the other part consists of the algorithms, or rules, for completing mathematical tasks"*.

Many secondary school learners are prepared to treat mathematics procedurally where they have to learn rules and use them in a certain way without fully understanding the "why" of the procedure. University lecturers often complain that first year students have little understanding of the basic concepts of pre-calculus and that even the better students are only equipped to deal with procedural problems rather than problems that need conceptual understanding (Engelbrecht *et al*., 2005:701). Although the use of procedures is an important part of mathematics and the proficiency of using computational tools is necessary, it is possible that procedural knowledge can be achieved without understanding, and someone who has only procedural knowledge may not be able to solve real-life problems and problems in different contexts.

Conceptual knowledge is characterized as knowledge that is rich in relationships (Hiebert & Lefevre, 1986:3-4). It is constructed internally and exists in the mind as part of a network of ideas (Van de Walle, 2007:28). It is knowledge that is understood and is not stored as isolated pieces of information (Hiebert & Carpenter, 1992:78). The NYSED (2009) indicates that conceptual understanding "*consists of those relationships constructed internally and connected to already existing ideas. It involves the understanding of mathematical ideas and procedures and includes the knowledge of basic arithmetic facts. Students use conceptual knowledge when they identify and apply principles, know and apply facts and definitions, and compare and contrast related concepts. Knowledge learned with understanding provides a foundation for remembering or reconstructing mathematical facts and methods, for solving new and unfamiliar problems, and for generating new knowledge"*.

Studies done by Gamoran (2001), Hiebert (2003) and the National Research Council (2001), as indicated by Hiebert and Grouws (2007:383), show that students can acquire conceptual knowledge if teaching attends explicitly to connections among mathematical facts, procedures and ideas. Mathematical connections should be treated in an explicit way because relationships between pieces of information do not always occur spontaneously (Hiebert & Lefevre, 1986:17). Teaching should include the discussion of the mathematical meaning underlying procedures. Unpacking these procedures could lead to understanding of a concept (Plotz, 2007:55). For example: Students may know how to use the laws of differentiation to determine $f'(x)$ if $f(x) = -2x^2$, but they will only have a relational understanding of mathematics if they can connect the process to the concept of a slope of a tangent line.

Hiebert and Lefevre (1986:16) state that being competent in mathematics involves knowing concepts, symbols and procedures and knowing how they are related. From their point of view both procedural and conceptual knowledge is important as students are not fully competent in mathematics if either kind of knowledge is deficient, or if they have been acquired but remain separate entities. If students only have instrumental understanding of procedures it refers to rote performance of procedures, so that they would not be able to transfer their procedural knowledge to new situations, nor to solve novel problems (Silver, 1986:185). Silver emphasizes that pure forms of procedural and conceptual knowledge are seldom exhibited. The relationship between these types of knowledge helps a person to apply his/her knowledge in a wide variety of settings. In the following example, understanding of the concept of a derivative is necessary, and procedural fluency will not be sufficient to solve the problem:

> *If $y = 2x + p$ is the equation of a tangent to $f(x) = -2x^2$, determine the coordinates of the point where the tangent touches the graph of $f(x)$.*

In this case the student should have a relational understanding of the slope of a tangent line to a curve as the derivative of the function of the curve. Gray and Tall (1994:129) observed that some children cling to the security of known step-by-step procedures, while others compress their knowledge into thinkable concepts for more sophisticated thinking (Tall, 2007:4).

It is generally accepted that procedures should not be taught in the absence of concepts. If procedures are taught without a conceptual base, the result is rules without reasons. All mathematics procedures should be connected to the conceptual ideas that explain why they work (Van de Walle, 2007:28). Students should build internal representations of procedures that become part of larger conceptual networks before they are encouraged by the repeated practice of procedures (Hiebert & Carpenter, 1992:79).

3.5 The cognitive processes through which mathematical understanding develops

3.5.1 Representations

In order to think about mathematical ideas, one needs to represent them in some way. Representations can be internal or external (Pape & Tchoshanov, 2001:119; Hiebert & Carpenter, 1992:66). Pape and Tchoshanov see the abstractions of mathematical ideas that are developed by a learner through experience as internal representations and the representation of numerals, algebraic equations, graphs, diagrams and charts as external manifestations of mathematical concepts. Therefore representation refers to the act of internalizing an internal, mental abstraction (Lesser & Tchoshanov, 2005:1).

In their research, Lesh *et al*. (1987:34) identified five types of representation systems that occur in mathematics learning and problem solving. They see representations as "*external embodiments of students' internal conceptualizations*" and emphasize that the distinct types of representations are important, but that translations among them as well as transformations within them are also important.

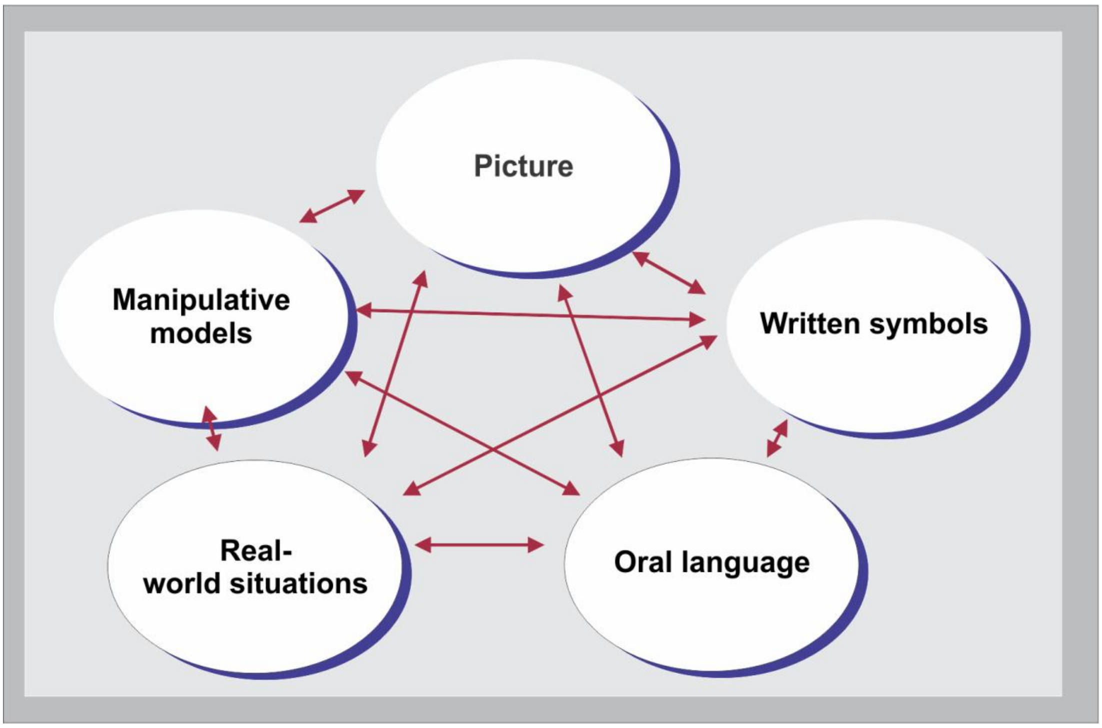


Fig 3.2: Five different representations of mathematical ideas (Lesh *et al.*, 1987:34)

Manipulative models – examples of these are number lines, fraction bars, countable objects, etc. These models refer to an object or drawing that represents the concept onto which the relationship for that concept can be imposed (Van de Walle, 2007:31). The concept of integers is often modelled with counters in two colours. The opposite aspect of the integers can be imposed on the two colours and the magnitude aspect is found in the quantities of the two colours. It is important to remember that physical models and drawings only represent mathematical concepts to the extent that the desired mathematical relationship can be imposed on them (Van de Walle, 2007:34).

Pictures or diagrams can be internalized as images. Tall (2004a:30) describes this as the "embodied world" where people think about the things around them in the physical world. It does not only include mental perceptions of real-world objects, but also people's internal conceptions that involve their own mental world of meaning. It is based on the perception of and reflection on properties and objects, patterns, verbal descriptions and definitions that formulate relationships and deductions.

Written symbols can involve specialized sentences and phrases such as $\mathrm{x} + 3 = 7$ or $A' \cup B' = (A \cap B)'$, as well as normal sentences and phrases. Tall (2010:22) explains that the "symbolic world" develops out of the embodied world through actions such as counting, sharing and measuring. It begins with actions such as pointing and counting and involves practising actions until they can be performed accurately with little conscious effort. It develops beyond the learning of procedures to carry out a process to the concept created by that process.

Oral language – language plays an important role in the learning of mathematics. Teachers communicate and explain and learners need to understand and respond and be able to communicate their understanding. Once students understand *how* things are said, they can better understand *what* is being said, and only then do they have a chance to know *why* it is said (Jamison, 2000:45). In advanced mathematics the focus should not only be on the understanding of the concept, but also on the formal mathematical syntax (Engelbrecht, 2010:143-146).

Real world situations – these are used to serve as general contexts for interpreting and solving other kinds of problem situations (Lesh *et al.*, 1987:33). Lesh *et al*. (1987:38) state that good problem solvers tend to be flexible in their use of relevant representational systems. They switch instinctively to the most convenient representation.

A useful way to describe understanding is in terms of how someone's internal representations are structured (Hiebert & Carpenter, 1992:66). For success in mathematics, students need to build internal representations of procedures that become part of bigger conceptual networks (Hiebert & Carpenter, 1992:79). Therefore the development of students' thinking skills requires a multiple representational approach (Pape & Tchoshanov, 2001:123). Once someone has formed meaning of different representations, he/she can use these representations in other situations. Moving from one representation to another is an important way of adding understanding to an idea (Van de Walle, 2007:5). Part of problem solving is to relate to different types of representations. The more different types of representations a student has, the more expanded his/her conceptual knowledge is (Plotz, 2007:67).

An example where different representations are used for better understanding of a mathematical concept is the concept of a function. The definition of a function, "*a rule that uniquely associates elements of one set with elements of another set*", may be too formal for learners to understand what a function is. The function concept evolves best from contextual situations where students can see that a change in one variable (independent variable) causes a change in another variable (the dependent variable). One can represent a function in five different ways (fig. 3.3), namely: the problem itself, referred to as the context; using language; in a table; as a graph and as an equation. Different representations of a function provide different ways of looking at or thinking about the function. The *context* provides an embodiment of the relationship outside the world of mathematics. *Language* helps to express the relationship in a meaningful and useful way. A function can be described by a *table* of values that match selected elements that are paired by the function. The *graph* translates the number pairs into a picture. The *equation* expresses the functional relationship between the elements.

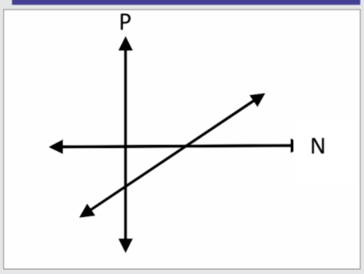
Language

The amount of profit that can be made by selling hot dogs is a function of the number of hot dogs sold.

Context

Selling hot dogs and the resulting profit.

Graph



Table

Hot dogs (N)	Profit (P)
0	-35
70	301
100	445
150	685

Equation

$$P = 4{,}8N - 35$$

Fig. 3.3 The five representations of a function (Van de Walle, 2007:280)

The different representations are all connected and illustrate the same relationship. Each representation provides a different perspective on the function (Van de Walle, 2007:280).

3.5.2 Connections

Understanding mathematics also relates to the quality and quantity of connections that new knowledge has with prior knowledge. When mathematics is understood, mathematical ideas build on one another in a useful network of connected ideas instead of only isolated rules and formulas. Mathematics plays a significant role in other disciplines and should be connected to the real world by means of applications in different *problem contexts* (Van de Walle, 2007:5). This fact is emphasized in the NCS (DoE, 2003:10): "*An important purpose of mathematics in the FET band is the establishment of proper connections between mathematics as a discipline and the application of mathematics in real-world contexts. Mathematical modelling provides learners with the means to analyse and describe their world mathematically, and so allows learners to deepen their understanding of mathematics".*

If teachers' beliefs are that mathematics is a set of rules and procedures that need to be followed with precision and mathematics is taught that way, then first year students will lack a deeper understanding of mathematics. Procedural knowledge would then be the only knowledge that has been developed and none of the other processes needed for understanding would have been developed. This could have a huge impact on students' first encounter with mathematics on tertiary level where it is expected that they should be able to exhibit conceptual understanding, represent their understanding in different ways, reason with understanding and make connections in different problem contexts.

3.5.3 Reasoning

Mathematical reasoning is an important process through which mathematical understanding develops. According to Dossey *et al*. (2002:76), reasoning occurs when students are given opportunities to explore on their own and are expected to verify the results of their explorations. Reasoning is the logical thinking that helps us decide if and why our answers make sense (Van de Walle, 2007:5). Plotz (2007: 62) indicates that for students to make sense of mathematics, they need to be able to communicate their knowledge to others and reasoning is a way of communication.

3.5.3.1 Deductive and inductive reasoning

Inductive and deductive reasoning play complementary roles in mathematical reasoning (Morris, 2002:81). Morris indicates that conjectures are usually formulated through inductive reasoning and these conjectures are then proven by a deductive statement. For example:

To prove that for every counting number n, the expression $n^2 + n$ *will always be even, we can begin with an inductive proof to show that if n = 1 and n = 2 then the result is an even number. If this is true for an odd number as well as an even number, we can conclude that it will be true for all counting numbers. However this is not a valid proof. Only a deductive argument can establish that the relation holds for every element in the infinite set of counting numbers.*

Deductive reasoning is the logical foundation of mathematical proofs of theorems. Commonly accepted notions of deductive reasoning is the process of inferring conclusions from known information based on formal logic rules where conclusions are derived from the given information and there is no need to validate them by experiments (Ayalon & Even, 2010:1131). Deductive reasoning is deciding that a specific or a particular problem is subsumed by a generality. It is the cognitive process along which people determine whether what they know about a concept or abstract relationship is applicable to some specific situation (Cangelosi, 2003:255). Deductive reasoning is often synonymous with mathematical thinking. This type of reasoning corresponds to the static-formalist view of mathematics where mathematics is viewed as an axiomatic–deductive science (par. 2.5.1).

Also inherent in deductive reasoning is the use of syllogisms (Cangelosi, 2003:255). A syllogism is a scheme in which a conclusion is drawn from a major premise and a minor premise. The major premise is a general rule or abstraction and the minor premise is the relationship of the specific to the general rule or abstraction. The conclusion is a logical consequence of the combined premises.

An example:

Major premise: If the discriminant ($b^2 - 4ac$) of a quadratic equation ($ax^2 + bx + c = 0$) is positive and not a perfect square, the equation has two irrational roots.

Minor premise: The discriminant of $x^2 - x - 18 = 0$ is 73, which is positive and not a perfect square.

Conclusion: $x^2 - x - 18 = 0$ has two irrational roots.

According to Gonzàlez and Herbst (2006:8) geometry was traditionally used as a model for teaching deductive reasoning and one of the main goals for teaching geometry was to develop deductive reasoning. In the South African curriculum, the topics of absolute values and Euclidian geometry were initially excluded from the school curriculum (NCS). The higher order reasoning skills developed in these two topics are important for tertiary mathematics. In the South African school context geometry was an optional subject for the past years. Few learners take the optional geometry (Mathematics Paper III) and proofs are no longer necessary and part of the syllabus. It is obvious that there is little chance in secondary school for learners to develop their deductive reasoning skills. Therefore a gap exists between the reasoning necessary to be successful in secondary school mathematics and in tertiary mathematics.

This indicates that the gap caused by the exclusion of certain topics from the secondary school curriculum is not merely a knowledge gap, but the result of the exclusion is that deductive reasoning skills were not developed anymore in the secondary school.

The type of reasoning where patterns are observed and then answers are predicted for more complicated problems is called *inductive reasoning* (Smith, 2004:18). It involves reasoning from particular facts or cases to a general conjecture – a statement you think may be true. Thus a generalisation is made on the basis of some observed occurrences. The more occurrences one observes, the more one will be able to make a correct generalisation. This is in line with the relativist-dynamic view of mathematics where learners are actively involved in constructing meaning from experiences by searching for patterns while doing mathematics. Cangelosi (2003:177) defines inductive reasoning as the cognitive process by which people discover commonalities among specific examples, thus leading them to formulate abstract categories (i.e. concepts) or discover abstract relationships. Inductive reasoning is thus aimed at generalisations, rules or regularities from individual observations and experiences.

Klauer and Phye (2008:87) indicates that inductive reasoning consists of detecting regularities and irregularities by finding out the

$$\left\{\begin{matrix} similarity \\ difference \\ similarity\ \ and\ difference \end{matrix}\right\} \text{ of } \left\{\begin{matrix} attributes \\ relations \end{matrix}\right\} \text{ with } \left\{\begin{matrix} verbal \\ pictorial \\ geometrical \\ numerical \\ other \end{matrix}\right\} \text{ material.}$$

An example where inductive reasoning is used (Smith, 2004:19) is the following:

Problem: *What is the sum of the first 100 consecutive odd numbers?*

This means 1 + 3 + 5 +…………+ ?

What will the last term be so that you will know when you have reached 100 consecutive odd numbers?

Look at the pattern: *1 + 3 is two terms*

1 + 3 + 5 is three terms

1 + 3 + 5 + 7 is four terms

Thus the last term is always one less than twice the number of terms.

This means that the sum of 100 consecutive odd numbers will be:

1 + 3 + 5 + …+197 + 199

To determine the sum:

Sum of one term: *1 = 1*

Sum of two terms: *1 + 3 = 4*

Sum of three terms: *1 + 3 + 5 = 9*

Sum of four terms: $\quad$ *1 + 3 + 5 + 7 = 16*

It appears that the sum of two terms is 2 x 2,

of three terms is 3 x 3,

of four terms is 4 x 4, and so on.

Thus the sum of 100 consecutive odd terms will then be
100 x 100 = 10 000

Students/learners use inductive reasoning to discover relationships or to figure out why the relationships exist in order to construct conceptual understanding. Felder and Silverman (1988: 677) indicate that the "best" method of teaching mathematics is with induction because induction is the "*natural human learning style*". The following sentences extracted from the NCS (2003) indicate that learners from secondary schools should be able to reason inductively:

- *"Through investigations, produce conjectures and generalisations related to triangles…."* (DoE, 2003:32).
- *"Investigate number patterns and make conjectures and generalisations….."* (DoE, 2003:18).
- *"Investigate, generalise and apply the effect of the following transformations of the point (x;y)"* (DoE, 2003:34).

A learner/student has to have a solid knowledge base in order to be able to reason inductively. Mature mathematical reasoning requires implicit understanding of inductive conclusions, as well as the necessity and sufficiency of deductive arguments and the complementary roles played by each form of reasoning (Morris, 2002:81).

3.5.3.2 Imitative and creative reasoning

Mathematical reasoning is not only restricted to proof, but is the way of thinking to produce assertions and reach conclusions. It includes the thinking done when confronted with ordinary mathematics tasks (Lithner, 2006:4). Lithner proposes a framework for analysing mathematical reasoning (fig. 3.4). This framework is restricted to the analysis of mathematical task solving where a task is seen as exercises, problems and tests.

Imitative reasoning means copying or following a model or example without any attempt of originality (Boesen, 2006:20). This way of reasoning is characterized by a search for familiar examples, solutions and other forms of “masters” from where the whole or parts of the solution can be reproduced. It is often successful as teachers and learners seem to have mutual expectations. Hiebert (1999:12) argues that students learn what they are given the opportunity to learn. Teachers let their students/learners work on tasks that they expect them to manage and the students/learners use solution methods that mainly work in most cases. Students/learners seem to seldom meet other types of tasks than the familiar ones and therefore never get a chance to develop their own problem solving skills.

This is what Skemp (1978) refers to as “instrumental” understanding of mathematics - the ability of a student to use a set of rules to get a correct answer to a problem. It is only the mastering of a rule without any insight into the reasons that make it work. Imitative reasoning, the copying of solutions to routine tasks, requires from students/learners only a surface understanding. Therefore imitative reasoning is a product of instrumental understanding and cannot serve as a foundation to solve unfamiliar tasks. Lithner (2006) describes different types of imitative reasoning and divides it in two main classes, namely memorised reasoning and algorithmic reasoning.

Memorised reasoning comprises of recalling a method and answer by memory and the implementation consists of writing it down. Any part of the answer can be described without having considered the preceding part. An example of memorised reasoning is to recall every step of a proof. *Algorithmic reasoning comprises* of recalling an algorithm that will guarantee that a correct answer can be reached. After the algorithm is recalled, only a careless mistake can prevent the learner from reaching an answer. The difference between these two categories of reasoning is that a student performing memorised reasoning has memorised the solution completely, while one performing algorithmic reasoning recalls a solution procedure and then performs it.

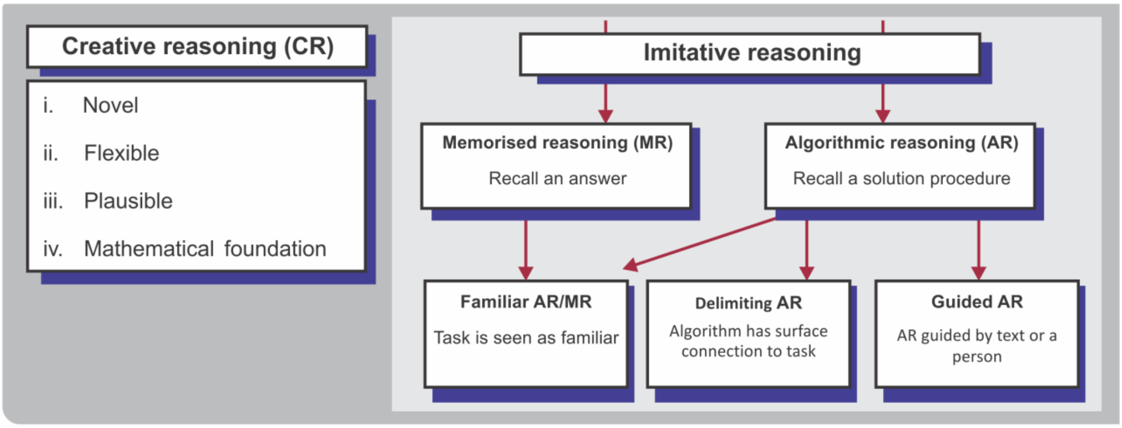


Fig 3.4 Reasoning types (Lithner, 2006:5)

There are three different versions of algorithmic reasoning (Lithner, 2006:5):

Familiar algorithmic reasoning is identified as being of a familiar type that can be solved by a known algorithm. The choice of strategy used is based on the experience that a task with certain textual, graphical and/or symbolical features is related to a corresponding algorithm. An example is when a student/learner connects a differentiation algorithm to a task asking for the maximum value of a function without considering the inherent meaning of differentiation.

Delimiting algorithmic reasoning is *w*hen a student chooses from a set of algorithms available and the set is delimited by choosing only algorithms that have some surface connection to the information in the task. The reasoner apprehends the task (wrongly or rightly) as a familiar one that corresponds to a known algorithm. For example: A function $f(x) = x^2 + 3$ is given and the question is to evaluate $\frac{f(x+h)-f(x)}{h}$. A student chooses to do $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. An algorithm has been selected from a set of algorithms that seemed to correspond with the surface property related to the task.

Guided algorithmic reasoning is when *a* person's reasoning can be guided by a source external to the task – through text or another person.

Text-guided reasoning is when a person identifies similarities between the task and an example, definition, theorem, rule, fact or some other situation in a text source. It is implemented by copying the procedure or fact from the identified situation. For example when a student looks for an example in the textbook and copies the procedure in every detail. An example (Lithner, 2006:19): Students had to find the largest or smallest values of the function $y = 7 + 3x - x^2$ on the interval [-1, 5]. A student looked for an example similar to the exercise – one with a second degree function with a maximum or minimum. The student copied the procedure and reached a solution, which he compared with the answers in the textbook's solution section without verification of the argumentation. This kind of textbook-guided reasoning is the dominating and sometimes the only type of student reasoning in learning situations (Lithner, 2006:19). In a previous study Lithner found that 70% of the exercises in the textbook could be completed by using text-guided reasoning (Lithner, 2004:422).

Person-guided reasoning is when someone (a teacher or a peer) pilots a student's solution. All strategy choices that could have been problematic for the solver are controlled and made by someone else. The teacher or the peer does not give predictive argumentation supporting these strategies. The strategy is carried through by following the guidance and executing routine transformations.

Example:

A student has difficulty in performing the task to determine the maximum value of a function $y = 7 + 3x - x^2$ in an interval [-1, 5]. The teacher then tells the student step by step what to do: "Differentiate the function. Then set this equal to zero and solve the equation. Evaluate by substituting back into y." There are no explanations on why one should differentiate, set equal to zero or substitute back. The student only executes routine procedures following the teachers' guidance without understanding.

Creative reasoning is reasoning that goes beyond just following strict algorithmic paths or recalling ideas provided by others (Lithner, 2006:5). It stems from a relational understanding, i.e. knowing what to do and why (Skemp, see par. 3.3). Relational understanding provides students/learners with conceptual structures that enable them to construct several plans for performing a given task. According to Silver (as noted by Lithner, 2006:7) creativity is related to deep, flexible knowledge in content domains and associated with long periods of work and reflection rather than rapid and exceptional insights. This view is reflected in the Curriculum and Assessment Policy Statement (CAPS) of Basic Education in South Africa (DoE, 2011). It states that the "*National Curriculum Statement for Grades R – 12 is based on active and critical learning, encouraging an active and critical approach to learning rather than rote and uncritical learning of given truths*".

Creativity is associated by the creation of new and well-founded task solutions. The characters of creative reasoning are novelty, flexibility, plausibility and mathematical foundation. *Novelty* is where a new sequence of reasoning is created or a forgotten sequence is re-created. There are no templates to follow. *Flexibility* means to admit different approaches and adaptations to the situation. *Plausibility* is used to describe reasoning that is supported by arguments that are not necessarily as strict as in proofs. There are arguments supporting the strategy choice, motivating why the conclusions are true. An example of this is inductive reasoning. Guesses, vague intuitions and affective reasons are not considered. There must be a *mathematical foundation* for the reasoning. It means that the argumentation is founded on mathematical properties of the components involved in the reasoning.

An example of creative reasoning (Lithner, 2006:10):

Find the largest and smallest values of the function $y = 7 + 3x - x^2$ on the interval [-1; 5].

A student first drew the graph on her calculator to see what it looked like. Her comments were: "I have seen that a x^2 -function can look like a valley and a hill. I can see that the minimum of this function is at the endpoint at $x = 5$. "She saw that the maximum seems to be where $x = 1.5$, but recalled that one cannot determine such a value from the graph. She calculated several values close to 1.5, but seemed unable to use them. She thought for 2 minutes. Then she said: "We have just learnt about derivatives….It says what a slope is. And the maximum is the only place where the slope is zero. I can do that, I think…..She calculated $y' = 3 - 2x$ and found its zero at $x = 1.5$ and evaluated $y(1.5) = 9.25$." She looked at her graph and said: "Yes it fits with the graph."

i) *Novelty: She has maximized second degree functions in earlier courses by completing the square, but forgot it. They just did derivatives, but not yet the maximisation algorithm. She constructed her own strategy and is not just following an algorithm.*

ii) *Flexibility: She was able to analyse the situation and adapt to its conditions, a type of initiative that is uncommon among students that focus on algorithmic approaches.*

iii) *Plausibility: She has mathematically based arguments for supporting the strategy choice and the conclusion.*

iv) *Mathematical foundation: She has a well-developed conceptual understanding of functions and based her reasoning on the relation between derivative, slope and maximum.*

As Lithner (2006:5) indicated, true mathematical proficiency goes further than just the mastering of basic facts and routine algorithms. A strong emphasis is placed on creativity when someone has to solve non-routine problems where reasoning goes beyond just following strict algorithmic paths or recalling ideas provided by others.

3.6 The gap due to different perceptions of the structure of mathematics

Success in secondary school mainly involves the application of various methods and carrying out certain procedures fluently (Engelbrecht, 2010:143). University lecturers expect from their students to have a higher degree of algebraic and numerical skills than in secondary school, as well as a deep conceptual understanding of mathematics to be able to increase the usefulness of the procedures practised in secondary school. The processes of “doing problems” at secondary school and that of “proving statements” at tertiary level are very different and require a change in reasoning processes. Students at universities must be able to work with rigorous definitions and theorems and apply rigid deductive reasoning. Brandell *et al*. (2008:48) found that first year students' background about proofs and deductive reasoning is very poor. Students had difficulties with exact mathematical language, as well as the mathematical symbols used in proofs because they found it different from the more "easier" language used in secondary school (Brandell et al., 2008:47). Nardy (as quoted by Hoyles *et al*., 2001:831) argues that most first year students have little idea of what mathematics at university level comprise – they assume that it may only be an extension of school mathematics and are therefore not prepared for the rigour and precision of university mathematics.

Many students at secondary school are only prepared to treat mathematics procedurally and instrumentally, i.e. to learn certain rules and use them in a standardized way (Engelbrecht *et al*., 2009:927). This reflects a general "surface" approach to learning. In Sweden Boesen (2006:3-4) found that secondary schools' emphasis is on computational skills. Most of the exercises done at school can be solved using a standard method imitated from previous work. Therefore learners do not get a real chance to develop their creative problem solving skills. This correlates with findings of Hourigan and O'Donoghue (2007) in Ireland, which showed that the focus in secondary schools is on the exams (mentioned or implied by action) just to obtain a good mark at the end of the year so that the students may be accepted for university studies. The behaviour of the teachers and the pupils were dictated by the fact that the leaving certificate is the only reason for studying mathematics. They found that comments like "This always comes up in the exam" and "You'll get good marks in the exam if…." were popular and a central component of each lesson. Hourigan and O'Donoghue reasoned that mathematics is taught without links to other relevant subjects and real world applications. Instead the focus is on the mastery of algorithmic procedures as isolated skills. They found that quick-fix approaches and drill work is done by the teachers and mathematics is about manipulating numbers and letters and filling in the right formula with no critical understanding. The only problem solving done is text-based story problems where the teacher would lead the pupils through the steps to achieve the right answer.

In the South African context the change from elite to mass education at tertiary institutions had an impact on the preparation for the exams in secondary schools. The reality of the examination at the end of the secondary phase emphasizes cognitive processes that require memorization of standard algorithms. Teachers coach their learners in order to get good marks in the final exams and to stand a chance to be selected for specific courses. No matter how much effort is put into creative learning, students are not interested as they can get high marks if they concentrate on memorizing of mathematical algorithms that were transmitted by their teachers. It is indicated that only 10 – 15% of the final grade 12 paper in South Africa expects of learners to do problem solving where conceptual understanding is a prerequisite (DoE, 2007:12). It can therefore be expected that students will struggle with problems that require creative reasoning.

In the next chapter the gap in the transition from elementary to advanced thinking will be investigated.