LIE SYMMETRY ANALYSIS OF A NONLINEAR BLACK-SCHOLES EQUATION OF MATHEMATICAL FINANCE

by

SENKEPENG LOUISA LEKALAKALA (16401182)

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Supervisor: Professor C M Khalique

Contents

Declaration

I, SENKEPENG LOUISA LEKALAKALA, student number 16401182, declare that this dissertation for the degree of Master of Science in Applied Mathematics at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

Ms SENKEPENG LOUISA LEKALAKALA

Date: ....

This dissertation has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Master of Science degree rules and regulations have been fulfilled.

Signed:..

PROF C.M. KHALIQUE

Date: ....

Declaration of Publications

Details of contribution to publications that form part of this dissertation.

Chapter 3

S. L. Lekalakala, T. Motsepa, C. M. Khalique, Lie symmetry reductions and exact solutions of an option-pricing equation for large agents. Submitted for publication to Mediterranean Journal of Mathematics

Dedication

To my father Edward Bokie Lekalakala and my late mother, Mmantsetse Deborah Lekalakala who passed on in July 2014.

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Abstract

The celebrated Black-Scholes equation is used to find the fair price of a financial instrument such as call and put options, the warrants and the down-and-out options as well as to find the implied volatility. Using Lie symmetry analysis we construct group-invariant solutions under some of the symmetry operators of the Black-Scholes equation. We then investigate the symmetry properties of the one-factor term structure option-pricing model by Jonsson and Keppo [Option Pricing for large Agents. Appl. Math. Finance, (2002) 9 261–272] which is a nonlinear modified Black-Scholes partial differential equation using Lie group analysis. An optimal system of one-dimensional subalgebras is derived which is used to obtain symmetry reductions and families of group-invariant solutions.

List of Acronyms

PDE: Partial differential equation

ODE: Ordinary differential equation

CIR: Cox-Ingersoll-Ross

BS: Black-Scholes

Introduction

The theory of option pricing began in 1900 when the French mathematician Louis Jean-Baptiste Alphonse Bachelier [1] deduced an option pricing formula based on the assumption that stock prices follow a Brownian motion.

Since the nineteen seventies many researchers, led by Black and Scholes, started using linear evolution equations to model the derivative security markets. The Black-Scholes equation

$$u_t + \frac{1}{2}x^2\sigma^2 u_{xx} + r\left(xu_x - u\right) = 0, \tag{1}$$

which does not cater for large traders was introduced by Black and Scholes [2] as the general equilibrium model of option pricing which is particularly attractive because the final formula is a function of observable variables. The assumptions which were used to derive this equation were considered too restrictive and unrealistic.

Equation (1) was also derived in [3] with weaker assumptions than those given in the original paper [2]. By introducing more assumptions, new explicit formulas were obtained for pricing both the call and put options as well as the warrants and the down-and-out options. Merton [3] was the first one to refer to equation (1) as the Black-Scholes equation. Equation (1) is sometimes referred to as the Black-Scholes-Merton equation and because of this work they were awarded the Nobel prize in economics, in 1997 even though Black did not receive it, as he passed away in 1995. The equation is mainly used to find the fair price of a financial instrument (derivative) and to find the implied volatility.

In the past few decades, a considerable amount of development has been made in symmetry methods for differential equations. This is evident by the number of research papers, books [4–15] and many new symbolic softwares [16–24] devoted to the subject.

Semi-invariants for the (1+1)-dimensional linear parabolic equations with two independent variables and one dependent variable were derived by Johnpillai and Mahomed [25]. In addition, the joint invariant equation was obtained for the linear parabolic equation and it was found that the (1+1)-dimensional linear parabolic equation was reducible via a local equivalence transformation to the one-dimensional heat equation. In [26], a necessary and sufficient condition for the parabolic equation to be reducible to the classical heat equation under the equivalence group was provided which improved on the work done in [25].

Goard [27] found group-invariant solutions of the bond-pricing equation by the use of classical Lie method. The solutions obtained showed that they satisfied the condition for the bond price, that is, $P(r, T) = 1$, where P is the price of the bond. Here r is the short-term interest rate which is governed by a stochastic differential equation and T is time to maturity.

Pooe *et al.* [28] obtained the fundamental solutions for a number of zero-coupon bond models by transforming the one-factor bond pricing equations corresponding to the bond models to the one-dimensional heat equation whose fundamental solution is well-known. Subsequently, the transformations were used to construct the fundamental solutions for zero-coupon bond pricing equations.

Sinkala *et al.* [29] computed the zero-coupon bonds (group-invariant solutions satisfying the terminal condition $u(T, T) = 1$) using symmetry analysis for the Vasicek and CIR equations, given by

$$u_t + \frac{1}{2}\sigma^2 u_{xx} + \kappa\left(\theta - x\right)u_x - xu = 0,$$

$$u_t + \frac{1}{2}\sigma^2 x u_{xx} + \kappa\left(\theta - x\right)u_x - xu = 0,$$

respectively. In [30] an optimal system of one-dimensional subalgebras was derived and used to construct distinct families of special closed-form solutions of CIR equation. In [31], group classification of the linear second-order parabolic partial differential equation

$$u_t + \frac{1}{2}\rho^2 x^{2\gamma} u_{xx} + (\alpha + \beta x - \lambda\rho x^{\gamma})u_x - xu = 0, \tag{2}$$

where α, β, λ, ρ and γ are constants was carried out. Lie point symmetries and group-invariant solutions were found for certain values of γ. Also the forms where the equation admitted the maximal seven Lie point symmetry algebra, (2) was transformed into the heat equation. Vasicek, CIR and Longstaff models were recovered from group classification and some other equations were derived which had not been considered before in the literature.

Dimas *et al.* [32] investigated some of the well known equations that arise in the mathematics of finance, such as Black-Scholes, Longsaff, Vasicek, CIR and Heath equations. Lie point symmetries of these equations were found and their algebras were compared with that of the heat equation. The equations with seven symmetries were transformed to the heat equation.

The assumption for the linear one-factor models applicable in liquid markets was that all traders are small, however, the assumption does not hold for illiquid markets which means that we need to have nonlinear models. In any illiquid market there are few interested buyers and hence most of the traders are large traders and their trading strategies determine market prices. In addition, big energy companies may affect the electricity price and any strategy trying to hedge their cash flow.

The general framework for securities' pricing in arbitrage-free complete markets without large agents is derived in [33–36]. The option replication for a large trader has been studied in [37–41] using nonlinear models.

In [37] the impact that derivative security markets have on market manipulation was investigated. Cvitanic and Ma [38] started by defining the large agent's impact function on the underlying price process and described option prices in terms of a forward-backward stochastic differential equation. The continuous-time version of Jarrow's model was studied in [39] and the existence and uniqueness of large agent hedging strategies for certain European options were proved. Option pricing under feedback effects from hedging was considered in [40, 41].

Jonsson and Keppo [42] presented a framework to value vanilla options, that is, European options and a model to estimate the parameters in their equation. Also, the same equation was derived for a single hedging agent and multiple hedging agents. A pricing game was considered which led to an equation for the number of outstanding options and it was shown that it was the same as determining the market shares of small and large agents. Numerical results were provided for option pricing in the presence of large and small agents, respectively. Recently a new partial Hamiltonian approach has been developed to deal with nonlinear models arising in mathematics of finance and economic growth theory [43].

In this dissertation, we study the nonlinear option pricing PDE

$$v_\tau - \frac{1}{2}s^2\sigma^2 e^{2av_s}v_{ss} - rsv_s + rv = 0, \tag{3}$$

which was first introduced in [42]. The parameter $\tau = T - t$ is time to maturity, s is the underlying stock, r is risk-free interest rate and σ is the volatility. The number a is a measure of the combined effect that the hedging activities of the agents have on the option prices and the option price v. This equation was first derived by assuming directly the form of the large agent effect on the underlying asset dynamics.

The aim of this work is to investigate the symmetry properties of the one-factor term structure option-pricing model by Jonsson and Keppo [42]. An optimal system of one-dimensional subalgebras is derived which is used to obtain symmetry reductions and family of group-invariant solutions.

The outline of this dissertation is as follows:

In Chapter one, the basic definitions and theorems concerning the one-parameter groups of transformations are presented.

In Chapter two, Lie symmetry method is employed to find symmetries of the Black-Scholes equation. The symmetries obtained are then used to compute group invariant solutions.

In Chapter three, we find Lie point symmetries of equation 3 and obtain optimal systems of one-dimensional subalgebras. We also obtain group-invariant solutions.

In Chapter four, a summary of the results of the dissertation is presented and future work is discussed.

Bibliography is given at the end of this dissertation.

Chapter 1

Lie symmetry analysis for partial differential equations

In this chapter, we present some salient features of Lie symmetry analysis of partial differential equations (PDEs). We also provide the algorithm to determine the Lie point symmetries of PDEs.

1.1 Introduction

Lie group analysis was developed by Sophus Lie (1842-1899) in the later half of the nineteenth century. He showed that the majority of adhoc methods of integration of differential equations could be explained and deduced simply by means of his theory. Recently, many books have appeared in literature on this subject. These are Ovsianikov [4, 5], Bluman and Kumei [6], Bluman and Anco [7], Stephani [8], Olver [9], Ibragimov [10–12], Hydon [13], Cantwell [14] and Bluman *et. al* [15].

Definitions and results given in this Chapter are taken from the books mentioned above.

1.2 Continuous one-parameter (local) Lie group

A transformation will be understood to mean an invertible transformation, that is, a bijective map. Let t and x be two independent variables and u be a dependent variable. Consider a change of the variables t, x and u given by

$$T_a : \bar{t} = f(t, x, u, a), \;\; \bar{x} = g(t, x, u, a), \;\; \bar{u} = h(t, x, u, a), \tag{1.1}$$

where a is a real parameter which continuously ranges in values from a neighbourhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$ and f, g and h are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables t, x and u if the following three conditions are satisfied:

(i) For $T_a, T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

From (i) it follows that the associativity property is satisfied. Also, if the identity transformation occurs at $a = a_0 \neq 0$, i.e., T_{a_0} is the identity, then a shift of the parameter $a = \bar{a} + a_0$ will give T_0 as above. The group property (i) can be written as

$$\begin{aligned} \bar{\bar{t}} &\equiv f(\bar{t}, \bar{x}, \bar{u}, b) = f(t, x, u, \phi(a, b)), \\ \bar{\bar{x}} &\equiv g(\bar{t}, \bar{x}, \bar{u}, b) = g(t, x, u, \phi(a, b)), \\ \bar{\bar{u}} &\equiv h(\bar{t}, \bar{x}, \bar{u}, b) = h(t, x, u, \phi(a, b)). \end{aligned} \tag{1.2}$$

The function ϕ is termed as the *group composition law.* A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial \phi(s, b)}{\partial b}\right|_{b=0}.$$

We now give the definition of a symmetry group for PDEs by considering, for example, evolutionary equations of the second-order, namely

$$u_t = F(t, x, u, u_x, u_{xx}), \quad \frac{\partial F}{\partial u_{xx}} \neq 0. \tag{1.3}$$

Definition 1.2 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.3) if (1.3) is form-invariant (has the same form) in the new variables $\bar{t}, \bar{x}$ and $\bar{u}$, i.e.,

$$\bar{u}_{\bar{t}} = F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}), \tag{1.4}$$

where the function F is the same as in equation (1.3).

1.3 Infinitesimal transformations

The Lie's theory states that, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations*:

$$\bar{t} \approx t + a\,\tau(t, x, u), \ \bar{x} \approx x + a\,\xi(t, x, u), \ \bar{u} \approx u + a\,\eta(t, x, u) \tag{1.5}$$

obtained from (1.1) by expanding the functions f, g and h into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f|_{a=0} = t, \quad g|_{a=0} = x, \quad h|_{a=0} = u.$$

Thus, we have

$$\tau(t,x,u) = \left.\frac{\partial f}{\partial a}\right|_{a=0}, \quad \xi(t,x,u) = \left.\frac{\partial g}{\partial a}\right|_{a=0}, \quad \eta(t,x,u) = \left.\frac{\partial h}{\partial a}\right|_{a=0}. \tag{1.6}$$

The vector (τ, ξ, η) with components (1.6) is the tangent vector at the point (t, x, u) to the surface curve described by the transformed points $(\bar{t}, \bar{x}, \bar{u})$, and is therefore called the *tangent vector field* of the group G.

We introduce the *symbol* of the infinitesimal transformations by writing (1.5) as

$$\bar{t} \approx (1 + a\,X)t, \quad \bar{x} \approx (1 + a\,X)x, \quad \bar{u} \approx (1 + a\,X)u,$$

where

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{1.7}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.3), we say that X is an *admitted operator* of (1.3) or X is an *infinitesimal symmetry* of equation (1.3).

1.4 Group invariants

Definition 1.3 A function $F(t,x,u)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{t},\bar{x},\bar{u}) \equiv F(f(t,x,u,a), g(t,x,u,a), h(t,x,u,a)) = F(t,x,u), \tag{1.8}$$

identically in t, x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(t,x,u)$ to be an invariant is that

$$X\,F \equiv \tau(t,x,u)\frac{\partial F}{\partial t} + \xi(t,x,u)\frac{\partial F}{\partial x} + \eta(t,x,u)\frac{\partial F}{\partial u} = 0. \tag{1.9}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has two functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(t,x,u) = c_1, \quad J_2(t,x,u) = c_2,$$

of the Lagrange's system

$$\frac{dt}{\tau(t,x,u)} = \frac{dx}{\xi(t,x,u)} = \frac{du}{\eta(t,x,u)}.$$

Theorem 1.3 Given the infinitesimal transformation (1.5) or its symbol X. the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{t}}{da} = \tau(\bar{t},\bar{x},\bar{u}), \quad \frac{d\bar{x}}{da} = \xi(\bar{t},\bar{x},\bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t},\bar{x},\bar{u}) \tag{1.10}$$

subject to the initial conditions

$$\bar{t}|_{a=0} = t, \quad \bar{x}|_{a=0} = x, \quad \bar{u}|_{a=0} = u\,.$$

1.5 Construction of a symmetry group

We now briefly describe the algorithm to determine a symmetry group for a given PDE. Firstly, we need to give some basic definitions.

1.5.1 Prolongation of point transformations

Consider a second-order PDE

$$E(t,x,u,u_t,u_x,u_{tt},u_{xx},u_{tx}) = 0, \tag{1.11}$$

where t and x are two independent variables and u is a dependent variable. Let

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{1.12}$$

be the infinitesimal generator of the one-parameter group G of transformation (1.1). The *first prolongation* of X is denoted by $X^{[1]}$ and is defined by

$$X^{[1]} = X + \zeta_1(t, x, u, u_t, u_x)\frac{\partial}{\partial u_t} + \zeta_2(t, x, u, u_t, u_x)\frac{\partial}{\partial u_x},$$

where

$$\begin{aligned} \zeta_1 &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\ \zeta_2 &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) \end{aligned}$$

and the total derivatives D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots. \quad (1.13)$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{tx}\frac{\partial}{\partial u_t} + \cdots. \quad (1.14)$$

Likewise, the second prolongation of X, denoted by $X^{[2]}$, is given by

$$X^{[2]} = X + \zeta_1\frac{\partial}{\partial u_t} + \zeta_2\frac{\partial}{\partial u_x} + \zeta_{11}\frac{\partial}{\partial u_{tt}} + \zeta_{12}\frac{\partial}{\partial u_{tx}} + \zeta_{22}\frac{\partial}{\partial u_{xx}}. \quad (1.15)$$

where

$$\begin{aligned} \zeta_{11} &= D_t(\zeta_1) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\ \zeta_{12} &= D_x(\zeta_1) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\ \zeta_{22} &= D_x(\zeta_2) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi). \end{aligned}$$

Using the definitions of D_t and D_x, one can write

$$\zeta_1 = \eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x\xi_u. \quad (1.16)$$

$$\zeta_2 = \eta_x + u_x\eta_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u. \quad (1.17)$$

$$\begin{aligned} \zeta_{11} = {} & \eta_{tt} + 2u_t\eta_{tu} + u_{tt}\eta_u + u_t^2\eta_{uu} - 2u_{tt}\tau_t - u_t\tau_{tt} - 2u_t^2\tau_{tu} - 3u_t u_{tt}\tau_u \\ & -u_t^3\tau_{uu} - 2u_{tx}\xi_t - u_x\xi_{tt} - 2u_t u_x\xi_{tu} - u_t^2 u_x\xi_{uu} \\ & -(u_x u_{tt} + 2u_t u_{tx})\xi_u. \end{aligned} \quad (1.18)$$

$$
\begin{aligned}
\zeta_{12} &= \eta_{tx} + u_x\eta_{tu} + u_t\eta_{xu} + u_{xt}\eta_u + u_t u_x \eta_{uu} - u_{tx}(\tau_t + \xi_x) - u_t\tau_{tx} \\
&\quad - u_{tt}\tau_x - u_t u_x(\tau_{tu} + \xi_{xu}) - u_t^2\tau_{xu} - (2u_t u_{tx} + u_x u_{tt})\tau_u - u_t^2 u_x \tau_{uu} \\
&\quad - u_x\xi_{tx} - u_{xx}\xi_t - u_x^2\xi_{tu} - (2u_x u_{tx} + u_t u_{xx})\xi_u - u_t u_x^2 \xi_{uu}. \qquad (1.19) \\
\zeta_{22} &= \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + u_x^2\eta_{uu} - 2u_{xx}\xi_x - u_x\xi_{xx} - 2u_x^2\xi_{xu} \\
&\quad - 3u_x u_{xx}\xi_u - u_x^3\xi_{uu} - 2u_{tx}\tau_x - u_t\tau_{xx} - 2u_t u_x\tau_{xu} - u_t u_x^2\tau_{uu} \\
&\quad - (u_t u_{xx} + 2u_x u_{tx})\tau_u. \qquad (1.20)
\end{aligned}
$$

1.5.2 Group admitted by a PDE

The operator

$$
X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}. \qquad (1.21)
$$

is said to be a (generator of) *point symmetry* of the second-order PDE

$$
E(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0 \qquad (1.22)
$$

if

$$
X^{[2]}(E) = 0 \qquad (1.23)
$$

whenever $E = 0$. This can also be written as (symmetry condition)

$$
X^{[2]}\, E\big|_{E=0} = 0. \qquad (1.24)
$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.4 Equation (1.24) is called the *determining equation* of (1.22), because it determines all the infinitesimal symmetries of equation (1.22).

The theorem below enables us to construct some solutions of (1.22) from the known ones.

Theorem 1.4 A symmetry of equation (1.22) transforms any solution of (1.22) into another solution of the same equation.

Proof: It follows from the fact that a symmetry of an equation leaves invariant that equation.

1.6 Lie algebras

Let X_1 and X_2 be any two operators defined by

$$X_1 = \tau_1(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \eta_1(t,x,u)\frac{\partial}{\partial u}$$

and

$$X_2 = \tau_2(t,x,u)\frac{\partial}{\partial t} + \xi_2(t,x,u)\frac{\partial}{\partial x} + \eta_2(t,x,u)\frac{\partial}{\partial u}.$$

Definition 1.5 (Commutator) The *commutator* of X_1 and X_2, written as $[X_1, X_2]$, is defined by the formula $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.6 (Lie algebra) A Lie algebra is a vector space L of operators such that, for all $X_1, X_2 \in L$, the commutator $[X_1, X_2] \in L$.

The dimension of a Lie algebra is the dimension of the vector space L.

It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \; [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

Theorem 1.5 The set of all solutions of any determining equation forms a Lie algebra.

1.7 Conclusion

In this chapter we gave a brief introduction to the Lie group analysis of PDEs and presented some results which will be used throughout this work. We also gave the algorithm to determine the Lie point symmetries of PDEs.

Chapter 2

Symmetry analysis of the Black-Scholes equation

In this chapter we consider the Black-Scholes (BS) equation, which arises in financial mathematics, and compute its symmetry Lie algebra. We also find group-invariant solutions under five symmetry generators of the BS equation. BS equation (1) was first investigated from the first perspective of Lie point symmetry analysis by Gazizov and Ibragimov [44], who found its symmetries and used two different transformations to transform it to the heat equation, which was used to solve the initial value problem. The invariance principle was used to construct the fundamental solution that could be used for general analysis of an arbitrary initial value problem.

Pooe *et al.* [45] obtained two classes of optimal systems of the one-dimensional subalgebras for the BS equation using the two transformations obtained by Gazizov and Ibragimov [44] that transformed BS to the heat equation. Sukhomlin and Ortiz [46] obtained solutions for the BS equation and the diffusion equation by ansatz using similarities between the two equations. Also in [46], the equivalence group for the BS equation was established and the largest set of transformations, each of which converts the BS equation to the diffusion equation was obtained.

In [47] two potential symmetries were found and used to obtain new solutions to the BS equation. First, the equation was written in conserved form which required the conservation laws. Conservation laws were found by the method of Kara and Mahomed [48], which uses symmetries to directly compute the conservation laws. Many other researchers also studied the BS equation from the point of view of Lie symmetry analysis and pricing of contingent claims. See for example [32, 49–53].

2.1 Lie point symmetries of the Black-Scholes equation

Consider the BS equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0. \tag{2.1}$$

where A, B and C are constants. This equation admits the one-parameter Lie group of transformations with infinitesimal generator

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{2.2}$$

if and only if

$$X^{[2]}(u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu)\bigg|_{(2.1)} = 0. \tag{2.3}$$

Using the definition of $X^{[2]}$ from the previous chapter, we obtain

$$\zeta_1 + A^2xu_{xx}\xi + \zeta_{22}\frac{1}{2}A^2x^2 + \xi Bu_x + Bx\zeta_2 - \eta C\bigg|_{(2.1)} = 0, \tag{2.4}$$

where ζ_1, ζ_2 and ζ_{22} are given by equations (1.16), (1.17) and (1.20) respectively. Substituting the values of ζ_1, ζ_2 and ζ_{22} in equation (2.4) (and replacing u_{xx} by

$\frac{2}{A^2x^2}\left[Cu - Bxu_x - u_t\right]$), we get

$$
\begin{aligned}
&\left[\eta_t + u_t\eta_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_tu_x\xi_u\right] + A^2\xi x\left[\frac{2}{A^2x^2}\left(Cu - Bxu_x - u_t\right)\right] + \\
&\frac{1}{2}A^2x^2\Bigg[\eta_{xx} + 2u_x\eta_{xu} + u_x^2\eta_{uu} - u_x\xi_{xx} - 2u_x^2\xi_{xu} - u_x^3\xi_{uu} - 2u_{tx}\tau_x - \\
&u_t\tau_{xx} - 2u_tu_x\tau_{xu} - 2u_xu_{tx}\tau_u - u_tu_x^2\tau_{uu} + \frac{2}{A^2x^2}\left(Cu - Bxu_x - u_t\right) \\
&\left(A^2x\xi + \eta_u - 2\xi_x - 3u_x\xi_u - u_t\tau_u\right)\Bigg] = 0. \qquad (2.5)
\end{aligned}
$$

Since τ. ξ and η are independent of the derivatives of u, we can split on the derivatives of u. This yields the following over determined system of linear PDEs:

$$
\begin{aligned}
u_{tx} &: \tau_x = 0, && (2.6)\\
u_xu_{tx} &: \tau_u = 0, && (2.7)\\
u_tu_x &: \xi_u = 0. && (2.8)\\
u_x^2 &: \eta_{uu} = 0. && (2.9)\\
u_t &: 2\xi_x - \frac{2}{x}\xi - \tau_t = 0. && (2.10)\\
u_x &: -\xi_t + Bx\xi_x - A^2x^2\eta_{xu} - \frac{A^2x^2}{2}\xi_{xx} - B\xi = 0. && (2.11)\\
1 &: \eta_t + Bx\eta_x - C\eta + \frac{A^2x^2}{2}\eta_{xx} + Cu\eta_u + \frac{2Cu}{x}\xi - 2Cu\xi_x = 0. && (2.12)
\end{aligned}
$$

Equations (2.6) and (2.7) imply that

$$\tau = a(t), \qquad (2.13)$$

where $a(t)$ is an arbitrary function of t. Equation (2.8) gives

$$\xi = b(t, x), \qquad (2.14)$$

where $b(t, x)$ is an arbitrary function of t and x. The integration of equation (2.9) twice with respect to u yields

$$\eta = c(t, x)u + d(t, x), \qquad (2.15)$$

where $c(t,x)$ and $d(t,x)$ are arbitrary functions of t and x. Substituting this value of ξ in (2.10), we obtain

$$b_x = \frac{1}{2}a'(t) + \frac{1}{x}b(t,x).$$

Solving the above equation for $b(t,x)$ we obtain

$$b(t,x) = \frac{1}{2}\,a'(t)x\ln x + xe(t), \tag{2.16}$$

where $e(t)$ is an arbitrary function of t. Thus

$$\xi = \frac{1}{2}\,a'(t)x\ln x + xe(t). \tag{2.17}$$

Substituting the values of ξ and η in (2.11), we obtain

$$c_x = \frac{1}{2xA^2}a''(t)\ln x + \frac{e'(t)}{xA^2} + \frac{a'(t)}{4x} - \frac{Ba'(t)}{2xA^2}. \tag{2.18}$$

Integrating the above equation with respect to x, we obtain

$$c(t,x) = \frac{a''(t)(\ln x)^2}{4A^2} - \frac{Ba'(t)(\ln x)}{2A^2} + \frac{e'(t)(\ln x)}{A^2} - \frac{a'(t)(\ln x)}{4} - f(t). \tag{2.19}$$

where $f(t)$ is an arbitrary function of t. Substituting the values of ξ and η in (2.12), we obtain

$$c_t u - d_t + Bx(c_x u - d_x) - Cd(t,x) + \frac{A^2x^2}{2}(c_{xx}u + d_{xx}) -$$

$$2Cu\left(\frac{a'(t)(\ln x)}{2} + e(t)\right) + Cua'(t)\ln x = 0. \tag{2.20}$$

Separating (2.20) with respect to u, we obtain

$$u \;:\; c_t + Bxc_x + \frac{A^2x^2c_{xx}}{2} - Ca'(t) = 0 \tag{2.21}$$

$$u^0 \;:\; d_t + Bxd_x - Cd(t,x) + \frac{A^2x^2d_{xx}}{2} = 0. \tag{2.22}$$

Substituting the value of c from (2.19) into (2.21), we obtain

$$\frac{a'''(t)(\ln x)^2}{4A^2} - \frac{Ba''(t)(\ln x)}{2A^2} + \frac{e''(t)\ln x}{A^2} + \frac{a''(t)\ln x}{4} + f'(t) +$$

$$Bx\left[\frac{a''(t)\ln x}{2A^2x} - \frac{Ba'(t)}{2A^2x} + \frac{e'(t)}{A^2x} + \frac{a'(t)}{4x}\right] + \frac{A^2x^2}{2}\left[\frac{a''(t)}{2A^2x^2} - \frac{a''(t)\ln x}{2A^2x^2} + \frac{Ba'(t)}{2A^2x^2} - \frac{e'(t)}{A^2x^2} - \frac{a'(t)}{4x^2}\right] - Ca'(t) = 0. \quad (2.23)$$

Separating (2.23) with respect to $\ln x$, we obtain

$$(\ln x)^2 \ : \ a'''(t) = 0 \quad (2.24)$$

$$(\ln x) \ : \ e''(t) = 0 \quad (2.25)$$

$$1 \ : \ f'(t) - \frac{B^2a'(t)}{2A^2} + \frac{Be'(t)}{A^2} + \frac{Ba'(t)}{2} + \frac{a''(t)}{4} - \frac{e'(t)}{2} - \frac{A^2a'(t)}{8} - Ca'(t) = 0. \quad (2.26)$$

Integrating (2.24) with respect to t three times, we obtain

$$a(t) = \frac{A_1t^2}{2} + A_2t + A_3. \quad (2.27)$$

Now integrating (2.25) gives

$$e(t) = A_4t + A_5. \quad (2.28)$$

Substituting the values of $a(t)$ and $e(t)$ into (2.26), and integrating gives

$$f(t) = \frac{B^2A_1t^2}{4A^2} + \frac{B^2A_2t}{2A^2} - \frac{BA_4t}{A^2} - \frac{BA_1t^2}{4} - \frac{BA_2t}{2} - \frac{A_1t}{4} + \frac{A_4t}{2} - \frac{A^2A_1t^2}{16} + \frac{A^2A_2t}{8} + \frac{CA_1t^2}{2} + CA_2t + A_6. \quad (2.29)$$

Substituting the values of $e(t)$, $a(t)$ and $f(t)$ into (2.19) we obtain

$$c(t,x) = \frac{A_1(\ln x)^2}{4A^2} - \frac{B(A_1t + A_2)\ln x}{2A^2} + \frac{A_4\ln x}{A^2} + \frac{(A_1t + A_2)\ln x}{4} + \frac{B^2A_1t^2}{4A^2} + \frac{B^2A_2t}{2A^2} - \frac{BA_4t}{A^2} - \frac{BA_1t^2}{4} - \frac{BA_2t}{2} - \frac{A_1t}{4} + \frac{A_4t}{2} + \frac{A^2A_1t^2}{16} + \frac{A^2A_2t}{8} + \frac{CA_1t^2}{2} + CA_2t + A_6. \quad (2.30)$$

Thus

$$\tau = \frac{1}{2}A_1t^2 + A_2t + A_3 \quad (2.31)$$

$$\xi = \frac{1}{2}(A_1 t + A_2) x \ln x + x(A_4 t + A_5) \tag{2.32}$$

$$\begin{aligned}\eta = \Bigg(& \frac{A_1(\ln x)^2}{4A^2} - \frac{B(A_1 t + A_2)\ln x}{2A^2} + \frac{A_4 \ln x}{A^2} + \frac{(A_1 t + A_2)\ln x}{4} \\ & + \frac{B^2 A_1 t^2}{4A^2} + \frac{B^2 A_2 t}{2A^2} - \frac{B A_4 t}{A^2} - \frac{B A_1 t^2}{4} - \frac{B A_2 t}{2} - \frac{A_1 t}{4} + \frac{A_4 t}{2} \\ & + \frac{A^2 A_1 t^2}{16} + \frac{A^2 A_2 t}{8} + \frac{C A_1 t^2}{2} + C A_2 t + A_6 \Bigg) u + d(t, x)\end{aligned} \tag{2.33}$$

and so the infinitesimal symmetries of the BS equation are

$$X_1 = \frac{\partial}{\partial t}, \tag{2.34}$$

$$X_2 = x\frac{\partial}{\partial x}, \tag{2.35}$$

$$X_3 = 2A^2 t\frac{\partial}{\partial t} + A^2 x \ln x\frac{\partial}{\partial x} + \left(2A^2 Ct + D^2 t - D\ln x\right) u\frac{\partial}{\partial u}, \tag{2.36}$$

$$X_4 = A^2 tx\frac{\partial}{\partial x} + (\ln x - Dt)u\frac{\partial}{\partial u}, \tag{2.37}$$

$$X_5 = 2A^2 t^2\frac{\partial}{\partial t} + 2A^2 tx \ln x\frac{\partial}{\partial x} + \left(2A^2 Ct^2 - A^2 t + (\ln x - Dt)^2\right) u\frac{\partial}{\partial u}, \tag{2.38}$$

$$X_6 = u\frac{\partial}{\partial u} \quad \text{and} \tag{2.39}$$

$$X_d = d(t, x)\frac{\partial}{\partial u}. \tag{2.40}$$

where $D = B - A^2/2$ and d is an arbitrary solution of (2.1). Furthermore, $X_1, \cdots, X_6$ are operators which generate six parameter group and X_d generates an infinite group.

2.2 Invariant solutions of the BS equation

In this section we construct group-invariant solutions under some of the symmetry operators of the BS equation. We start with the operator X_1.

Case 2.1 Let us calculate the invariant solution under the symmetry operator X_1. The operator X_1 is given by

$$X_1 = \frac{\partial}{\partial t}.$$

The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0},$$

which provide the two invariants $J_1 = x$ and $J_2 = u$. Thus, the invariant solution is given by $J_2 = \phi(J_1)$, that is,

$$u = \phi(x).$$

Substituting this value of u in (2.1), we obtain the following Cauchy-Euler ODE

$$\frac{1}{2}A^2x^2\phi'' + Bx\phi' - C\phi = 0.$$

The solution of this Cauchy-Euler equation is given by

$$\phi(x) = x^{\frac{A^2-2B-\sqrt{(A^2-2B)^2+8A^2C}}{2A^2}}\left(K_1x^{\frac{\sqrt{(A^2-2B)^2+8A^2C}}{A^2}} + K_2\right), \qquad (2.41)$$

where K_1 and K_2 are arbitrary constants. Hence the invariant solution of (2.1) under X_1 is

$$u(t,x) = x^{\frac{A^2-2B-\sqrt{(A^2-2B)^2+8A^2C}}{2A^2}}\left(K_1x^{\frac{\sqrt{(A^2-2B)^2+8A^2C}}{A^2}} + K_2\right). \qquad (2.42)$$

Case 2.2 We now obtain the invariant solution under the symmetry operator $X_2 = x\frac{\partial}{\partial x}$.

The characteristic equations are

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{0},$$

which provide the two invariants $J_1 = t$ and $J_2 = u$. Thus the invariant solution is given by $J_2 = \phi(J_1)$, i.e.,

$$u = \phi(t).$$

Substituting this value of u in (2.1), we obtain

$$\phi' - C\phi = 0.$$

Thus the second-order Black-Scholes PDE (2.1) reduces to first-order ODE

$$\frac{d\phi}{dt} - C\phi = 0.$$

Solving the above variables separable equation, we obtain

$$\phi(t) = K\exp(Ct),$$

where K is an arbitrary constant of integration. Hence the invariant solution of (2.1) is given by

$$u(t, x) = K\exp(Ct).$$

Case 2.3 Let us now construct an invariant solution under the symmetry generator

$$X_3 = 2A^2t\frac{\partial}{\partial t} + A^2x\ln x\frac{\partial}{\partial x} + (2A^2Ct + D^2t - D\ln x)u\frac{\partial}{\partial u}.$$

where $D = B - A^2/2$.

The characteristic equations

$$\frac{dt}{2A^2t} = \frac{dx}{A^2x\ln x} = \frac{du}{(2A^2Ct + D^2t - D\ln x)u}$$

provide the two invariants $J_1 = \ln x/\sqrt{t}$ and $J_2 = t\left(\frac{D^2}{2A^2} + C\right) - \frac{D\ln x}{A^2}$. Thus the invariant solution is given by $J_2 = \phi(J_1)$, i.e.,

$$u = \left\{t\left(\frac{D^2}{2A^2} + C\right) - \frac{D\ln x}{A^2}\right\}\phi\left(\frac{\ln x}{\sqrt{t}}\right).$$

Substituting this value of u in (2.1), we obtain

$$A^2\phi'' - z\phi' = 0 \text{ where } z = \frac{\ln x}{\sqrt{t}}.$$

The solution of the above second-order ODE is given by

$$\phi(z) = \sqrt{\frac{\pi}{2}}Ac_1 \operatorname{Erfi}\left(\frac{z}{A\sqrt{2}}\right) + c_2$$

where c_1 and c_2 are arbitrary constants and $\mathrm{Erfi}(z)$ denotes the imaginary error function $\mathrm{Erf}(iz)/i$ [54]. Hence the invariant solution of (2.1) under X_3 is given by

$$u(t,x) = \left\{t\left(\frac{D^2}{2A^2}+C\right) - \frac{D\ln x}{A^2}\right\}\left(\sqrt{\frac{\pi}{2}}Ac_1\ \mathrm{Erfi}\left(\frac{\ln x}{A\sqrt{2t}}\right)+c_2\right).$$

Case 2.4 Let us calculate the invariant solution under the operator X_4, namely

$$X_4 = A^2tx\frac{\partial}{\partial x} + (\ln x - Dt)u\frac{\partial}{\partial u},$$

where $D = B - A^2/2$.

Now

$$X_4 J \equiv 0 \quad \Rightarrow \quad A^2tx\frac{\partial J}{\partial x} + (\ln x - Dt)u\frac{\partial J}{\partial u} = 0. \tag{2.43}$$

The characteristic equations of (2.43) are

$$\frac{dt}{0} = \frac{dx}{A^2tx} = \frac{du}{(\ln x - Dt)u}.$$

Thus, one invariant is $J_1 = t$. The other is obtained from the equation

$$\frac{dx}{A^2tx} = \frac{du}{(\ln x - Dt)u}.$$

and is given by $J_2 = u/\exp\left\{\dfrac{(\ln x - Dt)^2}{2A^2t}\right\}$.

Consequently, the invariant solution of (2.1) under X_4 is $J_2 = \phi(J_1)$, i.e.,

$$u = \exp\left\{\frac{(\ln x - Dt)^2}{2A^2t}\right\}\varphi(t), \tag{2.44}$$

where φ is an arbitrary function of t. Substituting (2.44) into equation (2.1), gives

$$\varphi' + \left(\frac{1}{2t} - C\right)\phi = 0.$$

This is a first-order variables separable equation and its solutions is given by

$$\phi(t) = \frac{K}{\sqrt{t}}\mathrm{e}^{Ct},$$

where K is an arbitrary constant, and hence the invariant solution of the BS equation under the operator X_4 is

$$u(t,x) = \frac{K}{\sqrt{t}} \exp\left\{\frac{(\ln x - Dt)^2}{2A^2t} + Ct\right\}.$$

Case 2.5 Let us find the invariant solution under the operator X_5, namely

$$X_5 = 2A^2t^2\frac{\partial}{\partial t} + 2A^2tx\ln x\frac{\partial}{\partial x} + \left[(\ln x - Dt)^2 + 2A^2Ct^2 - A^2t\right]u\frac{\partial}{\partial u}.$$

The characteristic equations are

$$\frac{dt}{2A^2t^2} = \frac{dx}{2A^2tx\ln x} = \frac{du}{\left[(\ln x - Dt)^2 + 2A^2Ct^2 - A^2t\right]u}.$$

By considering

$$\frac{dt}{2A^2t^2} = \frac{dx}{2A^2tx\ln x}$$

and integrating, we obtain one invariant as $J_1 = \dfrac{\ln x}{t}$. The other invariant is obtained from the equation

$$\frac{dt}{2A^2t^2} = \frac{du}{\left[(\ln x - Dt)^2 + 2A^2Ct^2 - A^2t\right]u}.$$

and is given by $J_2 = u\sqrt{t}/\exp\left\{\dfrac{(\ln x - Dt)^2}{2A^2t} + Ct\right\}$.

Consequently, the invariant solution under X_5 is $J_2 = \phi(J_1)$, i.e.,

$$u = \frac{1}{\sqrt{t}}\exp\left\{\frac{(\ln x - Dt)^2}{2A^2t} + Ct\right\}\phi\left(\frac{\ln x}{t}\right). \tag{2.45}$$

Substituting u, u_t, u_x and u_{xx} in (2.1) and simplifying yields

$$\phi' = 0. \tag{2.46}$$

Solving equation (2.46) we obtain $\phi(J_1) = K_1J_1+K_2$ where K_1 and K_2 are arbitrary constants of integration. Hence equation (2.45) becomes

$$u(t,x) = \left(K_1\frac{\ln x}{t^{3/2}} + \frac{K_2}{\sqrt{t}}\right)\exp\left\{\frac{(\ln x - Dt)^2}{2A^2t} + Ct\right\}.$$

Remark: We note that the operators X_6 and X_d do not provide invariant solutions.

2.3 Conclusion

In this chapter we obtained the symmetry Lie algebra for BS equation. This equation arises in the mathematics of finance. We then constructed group-invariant solutions under some infinitesimal generators of the BS equation.

Chapter 3

Lie symmetry analysis of a nonlinear Black-Scholes equation

3.1 Introduction

In this chapter we study a nonlinear option pricing PDE in the presence of large traders given by

$$v_\tau - \frac{1}{2}s^2\sigma^2 e^{2av_s} v_{ss} - rsv_s + rv = 0. \tag{3.1}$$

which was first introduced in [42].

We start by determining the Lie point symmetries of (3.1) and then use them to find a one-dimensional optimal system of subalgebras from its Lie bracket and the adjoint representation table. We then perform symmetry reductions and obtain group-invariant solutions through the use of elements of the optimal system.

This work has been submitted for publication [55].

3.2 Lie point symmetries of (3.1)

The Lie point symmetries for (3.1) are given by the vector field

$$X = \xi^1(\tau, s, v)\frac{\partial}{\partial \tau} + \xi^2(\tau, s, v)\frac{\partial}{\partial s} + \eta(\tau, s, v)\frac{\partial}{\partial v} \tag{3.2}$$

if and only if

$$X^{[2]}\left(v_\tau - \frac{1}{2}s^2\sigma^2 e^{2av_s}v_{ss} - rsv_s + rv\right)\bigg|_{(3.1)} = 0, \tag{3.3}$$

where

$$X^{[2]} = X + \zeta_1\frac{\partial}{\partial v_\tau} + \zeta_2\frac{\partial}{\partial v_s} + \zeta_{22}\frac{\partial}{\partial v_{ss}}.$$

Here ζ_i's are given by

$$\zeta_1 = D_\tau(\eta) - v_\tau D_\tau(\xi^1) - v_s D_\tau(\xi^2),$$
$$\zeta_2 = D_s(\eta) - v_\tau D_s(\xi^1) - v_s D_s(\xi^2),$$
$$\zeta_{22} = D_s(\zeta_2) - v_{\tau s} D_s(\xi^1) - v_{ss} D_s(\xi^2).$$

where the total derivatives D_τ and D_s are defined as

$$D_\tau = \frac{\partial}{\partial \tau} + v_\tau\frac{\partial}{\partial v} + v_{\tau s}\frac{\partial}{\partial v_s} + v_{\tau\tau}\frac{\partial}{\partial v_\tau} + \cdots,$$
$$D_s = \frac{\partial}{\partial s} + v_s\frac{\partial}{\partial v} + v_{ss}\frac{\partial}{\partial v_s} + v_{\tau s}\frac{\partial}{\partial v_\tau} + \cdots.$$

Expanding equation (3.3) and replacing v_τ by $\frac{1}{2}s^2\sigma^2 e^{2av_s}v_{ss} + rsv_s - rv$, we obtain

$$s^4\sigma^4 v_{ss}^2 e^{4av_s}\left(\xi^1_s + v_s\xi^1_v\right) + 4s^2\sigma^2 e^{2av_s}\left(\xi^1_s + v_s\xi^1_v\right)v_{\tau s}$$
$$+ 2s^2\sigma^2 v_s v_{ss} e^{2av_s}\left(s^2\sigma^2 e^{2av_s}\xi^1_{sv} + 2ars\xi^1_s - 2arv\xi^1_v + 2a\xi^2_s - 2a\eta_v + 2\xi^2_v\right)$$
$$+ s^2\sigma^2 v_s^2 v_{ss} e^{2av_s}\left(s^2\sigma^2\xi^1_{vv}e^{2av_s} + 4ars\xi^1_v + 4a\xi^2_v\right)$$
$$+ s\sigma^2 v_{ss} e^{2av_s}\left(\sigma^2 s^3\xi^1_{ss}e^{2av_s} - 4arsv\xi^1_s - 4as\eta_s - 4\xi^2 + 2rs^2\xi^1_s + 2rsv\xi^1_v\right.$$
$$\left.-2s\xi^1_\tau + 4s\xi^2_s\right) + 2s^2\sigma^2 v_s^2 e^{2av_s}\left(2rs\xi^1_{sv} - rv\xi^1_{vv} + 2\xi^2_{sv} - \eta_{vv}\right)$$
$$+ 2s^2\sigma^2 v_s^3 e^{2av_s}\left(rs\xi^1_{vv} + \xi^2_{vv}\right) + 2s^2\sigma^2 v_s e^{2av_s}\left(rs\xi^1_{ss} - 2rv\xi^1_{sv} + \xi^2_{ss} - 2\eta_{sv}\right)$$

$$+ 4v_s\left(-\xi^2_\tau + r^2s^2\xi^1_s + r^2sv\xi^1_v - \xi^2 r - rs\xi^1_\tau + rs\xi^2_s + rv\xi^2_v\right)$$
$$- 2s^2\sigma^2e^{2av_s}\left(rv\xi^1_{ss} + \eta_{ss}\right) - 4r^2sv\xi^1_s - 4r^2v^2\xi^1_v - 4rs\eta_s + 4rv\xi^1_\tau - 4rv\eta_v$$
$$+ 4r\eta + 4\eta_\tau = 0$$

On separating the above equation on derivatives of v we get an overdetermined system of linear PDEs

$$v^2_{ss} : \xi^1_s = 0,$$
$$v^2_{ss}v_s : \xi^1_v = 0,$$
$$v_{\tau s} : \xi^1_s = 0,$$
$$v_{\tau s}v_s : \xi^1_v = 0,$$
$$v_sv_{ss} : s^2\sigma^2e^{2av_s}\xi^1_{sv} + 2ars\xi^1_s - 2arv\xi^1_v + 2a\xi^2_s - 2a\eta_v - 2\xi^2_v = 0,$$
$$v^2_sv_{ss} : s^2\sigma^2\xi^1_{vv}e^{2av_s} + 4ars\xi^1_v + 4a\xi^2_v = 0,$$
$$v_{ss} : \sigma^2s^3\xi^1_{ss}e^{2av_s} - 4arsv\xi^1_s - 4as\eta_s - 4\xi^2 + 2rs^2\xi^1_s + 2rsv\xi^1_v - 2s\xi^1_\tau + 4s\xi^2_s = 0,$$
$$v^2_s : 2rs\xi^1_{sv} - rv\xi^1_{vv} - 2\xi^2_{sv} - \eta_{vv} = 0,$$
$$v^3_s : rs\xi^1_{vv} + \xi^2_{vv} = 0,$$
$$v_s : 2s^2\sigma^2e^{2av_s}\left(rs\xi^1_{ss} - 2rv\xi^1_{sv} - \xi^2_{ss} - 2\eta_{sv}\right) + 4\left(-\xi^2_\tau - r^2s^2\xi^1_s + r^2sv\xi^1_v\right.$$
$$\left.-\xi^2 r - rs\xi^1_\tau + rs\xi^2_s + rv\xi^2_v\right) = 0,$$
$$1 : \; - 2s^2\sigma^2e^{2av_s}\left(rv\xi^1_{ss} + \eta_{ss}\right) - 4r^2sv\xi^1_s - 4r^2v^2\xi^1_v - 4rs\eta_s + 4rv\xi^1_\tau - 4rv\eta_v$$
$$+ 4r\eta + 4\eta_\tau = 0.$$

Simplifying the above system and further splitting on e^{v_s}, we obtain

$$\xi^1_s = 0, \tag{3.4}$$
$$\xi^1_v = 0, \tag{3.5}$$
$$\xi^2_v = 0, \tag{3.6}$$
$$\eta_{ss} = 0, \tag{3.7}$$

$$\eta_{vv} = 0, \tag{3.8}$$

$$\xi^2_{ss} - 2\eta_{sv} = 0, \tag{3.9}$$

$$\eta_v - {\xi^2}_s = 0, \tag{3.10}$$

$$2\xi^2 + s\left(2a\eta_s + \xi^1_\tau - 2{\xi^2}_s\right) = 0, \tag{3.11}$$

$$r\xi^2 + \xi^2_\tau - rs\xi^2_s + rs\xi^1_\tau = 0, \tag{3.12}$$

$$r\eta + \eta_\tau - rs\eta_s - rv\eta_v + rv\xi^1_\tau = 0. \tag{3.13}$$

To solve the above system of equations, we first observe from equations (3.4) and (3.5) that ξ^1 does not depend on s and v, which means that ξ^1 is a function of τ only. Thus

$$\xi^1 = \xi^1(\tau). \tag{3.14}$$

Equation (3.6) implies that ξ^2 depends on both τ and s but not on v. Thus

$$\xi^2 = \xi^2(\tau, s). \tag{3.15}$$

Integration of equation (3.7) with respect to s twice gives

$$\eta(\tau, s, v) = A(\tau, v)s + b(\tau, v), \tag{3.16}$$

where $A(\tau, v)$ and $b(\tau, v)$ are arbitrary functions of τ and v. Using the expressions for η and ξ^2 into (3.9) we obtain

$$\xi^2_{ss}(\tau, s) - 2A_v(\tau, v) = 0.$$

The above equation can be satisfied if and only if

$$\xi^2_{ss}(\tau, s) = c(\tau), \tag{3.17}$$

$$2A_v(\tau, v) = c(\tau), \tag{3.18}$$

where $c(\tau)$ is an arbitrary function of τ. Solving equations (3.17) and (3.18) we obtain

$$\xi^2(\tau, s) = \frac{s^2 c(\tau)}{2\sigma} + sj'(\tau) + e(\tau), \tag{3.19}$$

$$A(\tau, v) = \frac{vc(\tau)}{2\sigma} + d(\tau), \tag{3.20}$$

where $d(\tau)$, $e(\tau)$ and $f(\tau)$ are arbitrary functions of τ. Substituting (3.20) into (3.16) yields

$$\eta = \frac{vsc(\tau)}{2\sigma} + sd(\tau) + b(\tau, v). \tag{3.21}$$

Substituting (3.21) into (3.8) and integrating twice with respect to v we get

$$b(\tau, v) = vg(\tau) + h(\tau), \tag{3.22}$$

where $g(\tau)$ and $h(\tau)$ are arbitrary functions of τ. Substituting (3.22) into (3.21) we obtain

$$\eta = \frac{vsc(\tau)}{2\sigma} + sd(\tau) + vg(\tau) + h(\tau). \tag{3.23}$$

Substituting the expressions of η and ξ^2 into equation (3.10) and separating on s, we obtain

$$s: \; c(\tau) = 0, \tag{3.24}$$

$$1: \; f(\tau) = g(\tau). \tag{3.25}$$

The two above equations mean that

$$\xi^2 = sg(\tau) + e(\tau), \tag{3.26}$$

$$\eta = sd(\tau) + vg(\tau) + h(\tau). \tag{3.27}$$

Substituting (3.14), (3.26) and (3.27) into equation (3.11) we get

$$2asd(\tau) + 2e(\tau) + s\xi^{1\prime}(\tau) = 0.$$

After splitting the above equation on s we obtain

$$1: \; e(\tau) = 0, \tag{3.28}$$

$$s: \; 2ad(\tau) + \xi^{1\prime}(\tau) = 0. \tag{3.29}$$

Hence the new expression for ξ^2 is given by

$$\xi^2 = sg(\tau). \tag{3.30}$$

Using expressions (3.14) and (3.30) in equation (3.12) yields

$$g'(\tau) + r\xi'(\tau) = 0. \tag{3.31}$$

Integrating the above equation with respect to τ and making ξ^1 subject of formula we obtain

$$\xi^1(\tau) = c_1 - \frac{g(\tau)}{r} \tag{3.32}$$

where c_1 is an arbitration constant of integration.

Substituting equations (3.32), (3.30) and (3.27) into equation (3.13) we obtain

$$sd'(\tau) + h'(\tau) + rh(\tau) = 0.$$

The above equation gives the following two equations on splitting on s

$$s: \; d'(\tau) = 0, \tag{3.33}$$

$$1: \; h'(\tau) + rh(\tau) = 0. \tag{3.34}$$

Integration of equations (3.33) and (3.34) yield

$$h(\tau) = c_2 e^{-r\tau}, \tag{3.35}$$

$$d(\tau) = c_3, \tag{3.36}$$

where c_2 and c_3 are arbitrary constants of integration. Substituting (3.32) into (3.29) and integrating we get

$$g(\tau) = 2ac_3 r\tau + c_4, \tag{3.37}$$

where c_4 is an arbitrary constants of integration. Hence we get the following solution of equations (3.4) – (3.13):

$$\xi^1(\tau) = c_1 - 2ac_3\tau - \frac{c_4}{r}, \tag{3.38}$$

$$\xi^2(\tau, s) = 2ac_3rs\tau + c_4s, \tag{3.39}$$

$$\eta(\tau, s, v) = 2ac_3r\tau v + c_2e^{-r\tau} + c_3s + c_4v. \tag{3.40}$$

Thus the Lie algebra of infinitesimal symmetries of (3.1) is spanned by the four vector fields

$$\begin{aligned}
X_1 &= \partial_\tau, \\
X_2 &= e^{-r\tau}\partial_v, \\
X_3 &= -\frac{1}{r}\partial_\tau + s\partial_s + v\partial_v, \\
X_4 &= -(2a\tau)\partial_\tau + (2ars\tau)\partial_s + (2ar\tau v + s)\partial_v.
\end{aligned} \tag{3.41}$$

3.3 Optimal system of one-dimensional subalgebras of (3.1)

In this section we determine the set of equivalent families of group-invariant solutions from which all other solutions can be derived. Since there are too many combinations of symmetries to construct group-invariant solutions, it is not usually feasible to list all possible group-invariant solutions of a certain differential equation. We need an effective, systematic means of classifying these solutions, leading to an optimal system of group-invariant solutions [9].

In this regard we first need to find the optimal system of one-dimensional subalgebras. We start by computing the tables of Lie Brackets and the adjoint representation.

3.3.1 Computation of Lie Bracket

For any two symmetries X_i and X_j, its Lie bracket is given by $[X_i, X_j] = X_i(X_j) - X_j(X_i)$. The Lie Bracket has the properties that it is skew-symmetric, that is, $[X_i, X_j] = -[X_j, X_i]$ and that the diagonal elements in the Lie bracket table are all zero. The Lie Bracket table for the four Lie point symmetries of (3.1) is given in Table 3.1.

Table 3.1: The Lie Bracket table of Lie point symmetries

$[\,,]$	X_1	X_2	X_3	X_4
X_1	0	$-rX_2$	0	$2arX_3$
X_2	rX_2	0	0	0
X_3	0	0	0	$-2aX_3$
X_4	$-2arX_3$	0	$2aX_3$	0

3.3.2 Adjoint representation

To compute adjoint representations of symmetry operators (3.41) for equation (3.1), we use the result given in [9]

$$\begin{aligned}\text{Ad}(\exp(\varepsilon X_i))X_j &= \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(\text{ad}X_i)^n(X_j) \\ &= X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots .\end{aligned}$$

Thus, the entries of table of adjoints are tabulated in Table 3.2.

Table 3.2: Adjoint representation of Lie point symmetries

Ad	X_1	X_2	X_3	X_4
X_1	X_1	$e^{r\varepsilon}X_2$	X_3	$-2ar\varepsilon X_3 + X_4$
X_2	$X_1 - r\varepsilon X_2$	X_2	X_3	X_4
X_3	X_1	X_2	X_3	$2a\varepsilon X_3 + X_4$
X_4	$X_1 + r\left(1 - e^{-2a\varepsilon}\right)X_3$	X_2	$e^{-2a\varepsilon}X_3$	X_4

3.3.3 Optimal system of one-dimensional subalgebras

The Lie algebra L_4 spanned by the four Lie point symmetries provides a possibility to find invariant solutions of equation (3.1) which is based on an optimal system of one-dimensional subalgebras of L_4. In light of this we can write an arbitrary operator from L_4 as

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4. \tag{3.42}$$

which depends on the four arbitrary constants a_1, a_2, a_3, and a_4.

To construct the optimal system of one-dimensional subalgebras, we follow the method given in [9]. We act by adjoint map generated by X_2 and obtain

$$\begin{aligned}\bar{X} &= \mathrm{Ad}(e^{\varepsilon X_2})X \\ &= a_1\left(X_1 - r\varepsilon X_2\right) + a_2X_2 + a_3X_3 + a_4X_4 \\ &= a_1X_1 + \left(a_2 - a_1r\varepsilon\right)X_2 + a_3X_3 + a_4X_4.\end{aligned}$$

If $a_1 \neq 0$ and without loss of generality we choose $a_1 = 1$, we can set $\varepsilon = a_2/r$ which results in

$$\bar{X} = X_1 + a_3X_3 + a_4X_4.$$

Furthermore, if we assume $a_4 \neq 0$ and that $a_4 = 1$, then when we act on $\bar{X}$ by the adjoint map generated by X_3 we get

$$\begin{aligned}\tilde{X} &= \mathrm{Ad}(e^{\varepsilon X_3})\bar{X} = X_1 + a_3 X_3 + 2a\varepsilon X_3 + X_4 \\ &= X_1 + (a_3 + 2a\varepsilon)\, X_3 + X_4.\end{aligned}$$

We set $\varepsilon = -a_3/(2a)$ and obtain

$$\tilde{X} = X_1 + X_4,$$

which cannot be simplified further.

Now, if we assume $a_4 = 0$ and $a_1 = 1$ in (3.42) then we obtain $X = X_1 + a_2 X_2 + a_3 X_3$. The groups generated by X_2 and X_4 make the coefficients of X_2 and X_3 vanish, respectively. Hence, X is simplified to X_1.

We now consider the case when $a_1 = 0$ but $a_4 = 1$. The group generated by X_3 makes the coefficient of X_3 vanish and we get

$$X_4 - bX_2$$

which cannot be simplified further.

We now consider $a_1 = 0$ and $a_4 = 0$ in (3.42). This lead to

$$X = a_2 X_2 + a_3 X_3.$$

The group generated by X_1 when it acts on X gives

$$a_2 e^{r\varepsilon} X_2 + a_3 X_3. \tag{3.43}$$

If we assume $a_3 \neq 0$ and $a_3 = 1$, then we can choose ε such that $a_2 e^{r\varepsilon}$ is equal to $0, \pm 1$. Therefore, we get

$$X_3,\ X_3 + X_2,\ X_3 - X_2.$$

Lastly, we set $a_3 = 0$ in (3.43), which results in X_2.

Hence, the one-dimensional optimal system of subalgebras of (3.1) is given by

$\{X_1, X_2, X_3, X_3 + X_2, X_3 - X_2, X_1 + X_4, X_4 + bX_2\}$, where $b \in \mathbb{R}$.

The discrete symmetry $(\tau, x, v) \mapsto (\tau, -s, -v)$ will map $X_3 - X_2$ to $X_3 + X_2$, and hence the final optimal system of one-dimensional subalgebras is given by

$$\{X_1, X_2, X_3, X_3 + X_2, X_1 + X_4, X_4 + bX_2\}.$$

3.4 Symmetry reductions and group-invariant solutions

We use the optimal system of one-dimensional subalgebras computed in the previous section to construct group-invariant solutions of equation (3.1). This system of one-dimensional subalgebras gives six cases of group-invariant solutions. We note that operator X_2 does not have useful invariants, hence it is omitted when finding invariant solutions.

Case 1. X_1

The invariants are found from the solution of the associated Lagrange's system

$$\frac{d\tau}{1} = \frac{ds}{0} = \frac{dv}{0}$$

and are $J_1 = s$ and $J_2 = v$. Thus, the group-invariant solution of the equation (3.1) is given by $v = f(s)$, where $f(s)$ satisfies the nonlinear ODE

$$2rf(s) - 2rsf'(s) - s^2\sigma^2 e^{2af'(s)} f''(s) = 0. \tag{3.44}$$

Equation (3.44) has one Lie point symmetry

$$G = s\partial_s + f\partial_f.$$

The first-order invariants of G are used to reduce (3.44). These invariants are found from the solution of the associated Lagrange's system

$$\frac{ds}{s} = \frac{df}{f} = \frac{df'}{0}$$

and are $u = f/s$ and $w = f'$. Using these invariants the equation (3.44) is transformed to

$$\frac{dw}{du} = \frac{-2r}{\sigma^2 e^{2aw}}.$$

This is a variables separable equation whose solution is given by

$$e^{2aw} = -\frac{4ar}{\sigma^2}u + c_1,$$

where c_1 is a constant of integration. Reverting back to the original variables we obtain

$$e^{2av'} = -\frac{4arv}{s\sigma^2} + c_1$$

which can be reduced to the quadrature

$$s + c_2 = 2a \int \frac{dv}{\ln(c_1 - \frac{4arv}{\sigma^2 s})}.$$

Case 2. X_3

For this Lie point symmetry, the invariants are $J_1 = ve^{r\tau}$ and $J_2 = se^{r\tau}$ and hence the group-invariant solution is $J_1 = f(J_2)$. that is,

$$v = e^{-r\tau} f(se^{r\tau}). \tag{3.45}$$

Substituting (3.45) into (3.1), we obtain $f'' = 0$, whose solution is $f = c_1 se^{r\tau} + c_2$. Thus, the group-invariant solution under X_3 is

$$v = c_1 s + c_2 e^{-r\tau},$$

where c_1 and c_2 are constants of integration.

Case 3. $X_2 + X_3$

The two invariants of $X_2 + X_3$ are $J_1 = se^{r\tau}$ and $J_2 = r\tau + ve^{r\tau}$, and the group-invariant solution is $v = e^{-r\tau}\left[f(\zeta) - r\tau\right]$, where $\zeta = se^{r\tau}$ and f satisfies the second-order nonlinear ODE

$$2r + \zeta^2\sigma^2 e^{2af'(\zeta)} f''(\zeta) = 0. \tag{3.46}$$

The solution of equation (3.46) is given by

$$f(\zeta) = \frac{1}{2a}\left\{\zeta \ln\left(2a\left(c_1 + \frac{2r}{\zeta\sigma^2}\right)\right) + \frac{2r\ln\left(c_1\zeta\sigma^2 + 2r\right)}{c_1\sigma^2}\right\} + c_2 \tag{3.47}$$

and hence the group-invariant solution of (3.47) under the symmetry $X_2 + X_3$

$$v(\tau, s) = e^{-r\tau}\left(\frac{1}{2a}\left\{se^{r\tau}\ln\left[2a\left(c_1 + \frac{2re^{-r\tau}}{s\sigma^2}\right)\right] + \frac{2r\ln\left(c_1 s\sigma^2 e^{r\tau} + 2r\right)}{c_1\sigma^2}\right\} + c_2 - r\tau\right),$$

where c_1 and c_2 are arbitrary constants.

Case 4. $X_1 + X_4$

For the symmetry operator $X_1 + X_4$ we have the two invariants given by

$$J_1 = se^{r\tau}(1 - 2a\tau)^{\frac{r}{2a}} \text{ and } J_2 = \frac{1}{2a}\left\{e^{r\tau}(1 - 2a\tau)^{\frac{r}{2a}}(s\ln(1 - 2a\tau) + 2av)\right\}$$

The group-invariant solution of (3.1) is given by

$$v = \frac{1}{2a}\exp\left\{2ar\left(-\frac{\ln(1 - 2a\tau)}{4a^2} - \frac{\tau}{2a}\right)\right\}\left(2af(\zeta) - se^{r\tau}(1 - 2a\tau)^{\frac{r}{2a}}\ln(1 - 2a\tau)\right).$$

where $\zeta = se^{r\tau}(1 - 2a\tau)^{r/(2a)}$ and f satisfies the second-order nonlinear ODE

$$\zeta^2\sigma^2 f''(\zeta)e^{2af'(\zeta)} + 2r\zeta f'(\zeta) - 2\zeta - 2rf(\zeta) = 0. \tag{3.48}$$

The equation (3.48) has one Lie point symmetry which is given by

$$G = \zeta\partial_\zeta + f\partial_f.$$

The first-order invariants of G are used to reduce (3.44). These invariants are found from the solution of the associated Lagrange's system

$$\frac{d\zeta}{\zeta} = \frac{df}{f} = \frac{df'}{0}$$

and are $u = f/\zeta$ and $w = f'$. Using these invariants the equation (3.44) is transformed to

$$\frac{dw}{du} = \frac{2e^{-2aw}}{\sigma^2}\left(\frac{1}{(w-u)} - r\right).$$

Case 5. $X_4 + bX_2$

For the symmetry operator $X_4 + bX_2$ we get the two invariants

$$J_1 = se^{r\tau} \text{ and } J_2 = \frac{e^{r\tau}(s\ln(2ar\tau + b) + 2av)}{2a}.$$

The group-invariant solution of equation (3.1) is given by

$$v = e^{-r\tau} f(z) - \frac{s\ln(2ar\tau + b)}{2a}.$$

where $z = se^{r\tau}$ and $f(z)$ satisfies the second-order nonlinear ODE

$$\zeta\sigma^2 e^{2a\phi'(z)}\phi''(z) + 2r = 0.$$

The solution of the above equation is given by

$$\phi(z) = \frac{1}{2a}\left\{2az\ln\left(c_1 - \frac{2rz\ln z}{\sigma^2}\right) - e^{\frac{c_1\sigma^2}{2r}}\operatorname{Ei}\left(\ln z - \frac{\sigma^2 c_1}{2r}\right)\right\} + c_2,$$

where c_1 and c_2 are arbitrary constants and Ei is the exponential integral function [54]. Thus, the group-invariant solution of (3.1) under symmetry $X_4 + bX_2$ is

$$\begin{aligned} v(\tau, s) = &-\frac{s\ln(2ar\tau + b)}{2a} - \frac{e^{c_1\sigma^2/2r - r\tau}\operatorname{Ei}\left(\ln\left(e^{r\tau}s\right) - \sigma^2 c_1/2r\right)}{2a} \\ &+ \frac{s\ln\left[2a\left(c_1 - 2r\ln\left(se^{r\tau}\right)/\sigma^2\right)\right]}{2a} + c_2e^{-r\tau}. \end{aligned}$$

3.5 Conclusion

In this chapter, we showed that the nonlinear Black-Scholes PDE (3.1) admitted a four-dimensional Lie algebra. We used this four dimensional Lie algebra to compute the optimal system of one-dimensional subalgebras. With the help of the optimal system found, we performed symmetry reductions and constructed group-invariant solutions of equation (3.1). The solutions might have applications in finance and other fields of study.

Chapter 4

Concluding remarks

In recent years many phenomena in financial mathematics have been modelled by linear and nonlinear partial differential equations. The first mathematical model was provided by Black and Scholes and the partial differential equation is called Black-Scholes equation.

In this dissertation we first recalled some important definitions and results from Lie group analysis, which were later used in the dissertation.

In Chapter two we studied the Black-Scholes equation. We first obtained the Lie point symmetries of the Black-Scholes equation. We then used five symmetries of this equation and obtained five group-invariant solutions.

In Chapter three, we first obtained Lie point symmetries of the nonlinear Black-Scholes equation (3.1). We showed that the Lie algebra consisted of four symmetries. We then used these symmetries to determine an optimal system of one-dimensional subalgebras of (3.1). Finally, the group-invariant solutions of (3.1) were constructed based on the optimal system of one-dimensional subalgebras of (3.1).

In future work, I intend to compute group-invariant solutions of the nonlinear Black-Scholes equation (3.1) that would satisfy the terminal conditions.

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