APPROXIMATE GROUP CLASSIFICATION OF NON-LINEAR WAVE EQUATIONS

$$v_{tt} = \alpha e^{v_x} v_{xx} + \varepsilon g(v_x)$$

RJ MOITSHEKI

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by

RASEELO JOEL MOITSHEKI

Submitted in partial fulfilment of the requirements for the degree of Master of Science in the department of Mathematical Sciences in the faculty of Science and Technology at the University of North-West.

SUPERVISOR : Dr MT KAMBULE

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DECLARATION

I declare that the dissertation for the degree of Master of Science at the University of North-West hereby submitted, has not previously been submitted by me for a degree at this or any other university, that it is my own work in design and execution and that all material contained herein has been duly acknowledged.

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This dissertation entitled :

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Written by

Moitsheki RJ (91/06852)

of the department of Mathematical Sciences in the faculty of Science and Technology is hereby recommended for acceptance for examination

SUPERVISOR	:	Dr MT Kambule
DEPARTMENT	:	Mathematical Sciences
FACULTY	:	Science and Technology
UNIVERSITY	:	University of North-West

CONTENTS

ABSTRACT

We study the approximate group classification of a family of non-linear wave equations with a small perturbation. An essential part in this classification is the use of approximate equivalence transformations. We use these transformations to determine functions which extend the approximate principal Lie algebra. Furthermore we construct some invariant solutions.

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Chapter 1

Introduction

The problem of group classification of partial differential equations according to their symmetries, was first considered by Sophus Lie [12]. The algorithm for finding the symmetry group of a differential equation or systems of differential equations can be found in the literature, for example [7], [13], [15].

In the past two decades several papers, which are closely related to the present work, were published. To name a few, Ames et al [1] investigated the group properties of quasilinear hyperbolic equations of the form

$$v_{tt} = f\left(v_x\right) v_{xx}. \tag{1.1}$$

The investigation was later generalized by Torrisi et al [16], [17] to equations of the form

$$v_{tt} = f\left(x, v_x\right) v_{xx}$$

and further to the nonlinear wave equations of the form

$$v_{tt} = f\left(x, v_x\right) v_{xx} + g\left(x, v_x\right),$$

by Ibragimov, Torrisi and Valenti [10].

The method of classical group analysis of differential equations enables one to distinguish among all the equations of mathematical physics, the equations that are remarkable with respect to their symmetry properties. However, any perturbation of an equation destroys the group admitted and this in general reduces the practical values of these refined equations of group theoretic method [3].

This evoked the necessity for the development of approximate methods of group analysis suitable for the construction of symmetries which are stable with respect to a small perturbation of the equation. Various authors have done work in this area. To illustrate this point, we cite a few papers. Baikov, Gazizov and Ibragimov examined the approximate group properties of the second order ordinary differential equations of the form

$$u'' + u = \epsilon F(u),$$

Here $u = u(\theta)$, $u'' = \frac{d^2u}{d\theta^2}$ and $F(u) = \frac{-mf\left(\frac{1}{u}\right)}{L^2u^2}$, where L is angular momentum of a particle, m is the mass of the particle and u is the inverse of the distance from the center of a force, see Vol. 3 chapter 9 in [8]. They considered wave equations with a small dissipation, In particular they examined the sequence

$$w_{tt} + \epsilon w_t = F(w_{xx}) \overset{w_x=v}{\longrightarrow} v_{tt} + \epsilon v_t = f(v_x)\, v_{xx} \overset{v_x=u}{\longrightarrow} u_{tt} + \epsilon u_t = (f(u)\, u_x)_x \qquad (1.2)$$

which is connected by Bäcklund transformations. Here $f = F'$. These equations were generalized [6] to the equations of the form

$$u_{tt} + \epsilon\varphi(u)\, u_t = (f(u)\, u_x)_x\,.$$

Furthermore they [5] considered a class of evolution equations of the form

$$u_t = h(u)\, u_1 + \epsilon H,$$

where H is an arbitrary element of the space of differential functions. In one of their paper [6], they constructed approximate invariant solutions for the equation

$$u_{tt} + \epsilon u_t = (u^\sigma u_x)_x\,.$$

In chapter two, we will consider the group classification of the equations

$$v_{tt} = f(v_x)\, v_{xx}.$$

This was done by Baikov and Gazizov [5]. We do not claim originality of the work in this chapter but we merely provide the details of the classification result of the equations (1.1), in order to give a clear picture of the classification procedure for the subsequent sections. In fact we give a review of construction of exact symmetries of the unperturbed part of the nonlinear wave equation

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$$v_{tt} + \epsilon v_t = f(v_x)\, v_{xx}$$

given in the papers by Oron and Rosenau [14], Baikov and Gazizov [5]. In particular, we construct the principal Lie algebra, the equivalence transformations for the equation (1.1) and determine the functions for which the principal Lie algebra extends.

In chapter three we construct the principal Lie algebra and approximate equivalence transformations for the equations

$$v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon g(v_x) \tag{1.3}$$

and hence attempt to classify them.

In chapter four we construct the adjoint group for the 10-dimensional approximate principal Lie algebra L_{10}, we consider some linear combinations of their symmetries and we further construct some approximate invariant solutions for the equation

$$v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon A e^{\beta v_x}. \tag{1.4}$$

Finally, in appendices, we will give the prolongation formulae, definitions of the operators X, E and D_a (a is a variable) and we prove that the approximate point symmetries obtained in section (3.2) leave the equations (3.7) invariant.

Chapter 2

Group classification of the equations $v_{tt} = f(v_x) v_{xx}$

2.1 The Principal Lie Algebra

In this section we wish to determine the pointwise principal Lie algebra symmetries which are admitted by the family of equations (1.1). We shall denote this algebra by $L_{\wp}$. The local vector field on the (t, x, v) - space represented by

$$X = \xi^1(t,x,v)\frac{\partial}{\partial t} + \xi^2(t,x,v)\frac{\partial}{\partial x} + \eta(t,x,v)\frac{\partial}{\partial v}, \tag{2.1}$$

generates the elements of $L_{\wp}$. Our aim is to determine the functions ξ^1, ξ^2 and η in (2.1).

In the extended space with variables $(t, x, v, v_x, v_t, v_{xx}, v_{tt}, v_{xt})$, the prolonged operator becomes

$$X^{(2)} = X + \zeta_1 \frac{\partial}{\partial v_t} + \zeta_2 \frac{\partial}{\partial v_x} + \zeta_{11} \frac{\partial}{\partial v_{tt}} + \zeta_{22} \frac{\partial}{\partial v_{xx}} + \zeta_{12} \frac{\partial}{\partial v_{xt}}, \tag{2.2}$$

where[1]

$$\begin{aligned}
\zeta_1 &= D_t(\eta) - v_t D_t\left(\xi^1\right) - v_x D_t\left(\xi^2\right), \\
\zeta_2 &= D_x(\eta) - v_t D_x\left(\xi^1\right) - v_x D_x\left(\xi^2\right), \\
\zeta_{11} &= D_t(\zeta_1) - v_{tt} D_t\left(\xi^1\right) - v_{tx} D_t\left(\xi^2\right), \\
\zeta_{22} &= D_x(\zeta_2) - v_{xt} D_x\left(\xi^1\right) - v_{xx} D_x\left(\xi^2\right), \\
\zeta_{12} &= D_x(\zeta_1) - v_{xt} D_x\left(\xi^1\right) - v_{xx} D_x\left(\xi^2\right)
\end{aligned} \tag{2.3}$$

and the total *derivatives* D_t and D_x are given by

$$D_t = \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} + \ldots$$

$$D_x = \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v} + v_{xx} \frac{\partial}{\partial v_x} + v_{xt} \frac{\partial}{\partial v_t} + \ldots$$

The invariance condition on the equation (1.1)

$$X^{(2)}\left(v_{tt} - f(v_x)\, v_{xx}\right) |_{(1.1)} = 0, \tag{2.4}$$

yields the following determining equation

$$\zeta_{11} - f\zeta_{22} - f'\zeta_2 v_{xx} = 0. \tag{2.5}$$

In solving the equations (2.5), v_{tt} is replaced by $f(v_x)\, v_{xx}$ and v_t, v_x, v_{xx} and v_{xt} are considered as free variables. The decomposition of the equations (2.5) with respect to the free variables v_t, v_{xx} and v_{xt} leads to the equations

$$\xi^2_v = 0, \tag{2.6}$$

[1] The explicit prolongation formulae are given in appendix A

$$\xi_v^1 + f'\left(\xi_x^1 + \xi_v^1 v_x\right) = 0, \tag{2.7}$$

$$\xi_t^2 + f\left(\xi_x^1 + \xi_v^1 v_x\right) = 0, \tag{2.8}$$

$$\eta_{vv} - 2\xi_{vt}^1 = 0, \tag{2.9}$$

$$f\left(-2\xi_t^1 + 2\xi_x^2\right) = f'\left[\eta_x - \left(\eta_v - \xi_x^2\right) v_x\right], \tag{2.10}$$

$$\left(2\eta_{vt} - \xi_{tt}^1\right) + f\left(\xi_{xx}^1 + \xi_{xv}^1 v_x + \xi_{vv}^1 v_x^2\right) = 0, \tag{2.11}$$

$$\eta_{tt} - \xi_{tt}^2 - f\left[\eta_{xx} + \left(2\eta_{xv} - \xi_{xx}^2\right) v_x + \eta_{vv} v_x^2\right] = 0. \tag{2.12}$$

For an arbitrary function $f(v_x)$, all the coefficients should vanish, that is

$$\xi_v^1 = \xi_x^1 = \xi_v^2 = \xi_t^2 = \eta_{tt} = \eta_x = 0,$$

$$\xi_x^2 - \xi_t^1 = 0,$$

$$2\eta_{vt} - \xi_{tt}^1 = 0,$$

$$2\eta_{xv} - \xi_{xx}^2 = 0,$$

$$\eta_v - \xi_x^2 = 0,$$

$$\eta_{vv} - 2\xi_{xv}^2 = 0. \tag{2.13}$$

Solving the equations (2.13) we obtain

$$\xi^1 = c_1 t + c_2,$$

$$\xi^2 = c_1 x + c_3,$$

$$\eta = c_1 v + c_4 t + c_5, \tag{2.14}$$

where $c_1, ..., c_5$ are constants. Thus we obtain a 5-dimensional principal Lie algebra with basis

$$X_1 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v},$$

$$X_2 = \frac{\partial}{\partial t},$$

$$X_3 = \frac{\partial}{\partial x},$$

$$X_4 = \frac{\partial}{\partial v},$$

$$X_5 = t\frac{\partial}{\partial v}.$$

2.2 The equivalence transformations

In this section we wish to find all pointwise equivalence transformations of the equation (1.1).

Definition 1 *An equivalence transformation is a nondegenerate change of variables t, x and v, which takes any equation of the form (1.1) to an equation of the same form, generally with different coefficient $f(v_x)$ [3].*

We apply the Lie infinitesimal method to calculate the subgroup $\mathcal{E}_c$ of continuous transformations of the group of equivalence transformations $\mathcal{E}$ of the system

$$v_{tt} - f v_{xx} = 0,$$

$$f_x = f_t = f_v = f_{v_t} = 0.$$

We shall determine the operator

$$E = \xi^1(t,x,v)\frac{\partial}{\partial t} + \xi^2(t,x,v)\frac{\partial}{\partial x} + \eta(t,x,v)\frac{\partial}{\partial v} + \mu(t,x,v,v_x,v_t,f)\frac{\partial}{\partial f} \qquad (2.15)$$

which generates the elements of the subgroup $\mathcal{E}_c$, where t, x, v, v_x and v_t are independent variables and f is the only dependent variable.

Along with the operators

$$D_1 \equiv D_t = \frac{\partial}{\partial t} + v_t\frac{\partial}{\partial v} + v_{tt}\frac{\partial}{\partial v_t} + v_{xt}\frac{\partial}{\partial v_x} + ...,$$

$$D_2 \equiv D_x = \frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v} + v_{xx}\frac{\partial}{\partial v_x} + v_{xt}\frac{\partial}{\partial v_t} + ...,$$

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we introduce the following differential operators

$$\widetilde{D}_1 \equiv \widetilde{D}_t = \frac{\partial}{\partial t} + f_t\frac{\partial}{\partial f} + f_{tt}\frac{\partial}{\partial f_t} + f_{xt}\frac{\partial}{\partial f_x} + f_{tv}\frac{\partial}{\partial f_v} + f_{tv_t}\frac{\partial}{\partial f_{v_t}} + ...,$$

$$\widetilde{D}_2 \equiv \widetilde{D}_x = \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial f} + f_{xx}\frac{\partial}{\partial f_x} + f_{xt}\frac{\partial}{\partial f_t} + f_{xv}\frac{\partial}{\partial f_v} + f_{xv_t}\frac{\partial}{\partial f_{v_t}} + ...,$$

$$\widetilde{D}_3 \equiv \widetilde{D}_v = \frac{\partial}{\partial v} + f_v\frac{\partial}{\partial f} + f_{vv}\frac{\partial}{\partial f_v} + f_{xv}\frac{\partial}{\partial f_x} + f_{tv}\frac{\partial}{\partial f_t} + f_{vv_t}\frac{\partial}{\partial f_{v_t}} + ...,$$

$$\widetilde{D}_4 \equiv \widetilde{D}_{v_t} = \frac{\partial}{\partial v_t} + f_{v_t}\frac{\partial}{\partial f} + f_{v_tv_t}\frac{\partial}{\partial f_{v_t}} + f_{xv_t}\frac{\partial}{\partial f_x} + f_{tv_t}\frac{\partial}{\partial f_t} + f_{vv_t}\frac{\partial}{\partial f_v} + ...$$

In the extended space the operator (2.15) is given by

$$\widehat{E} = E + \zeta_1\frac{\partial}{\partial v_t} + \zeta_2\frac{\partial}{\partial v_x} + \zeta_{11}\frac{\partial}{\partial v_{tt}} + \zeta_{22}\frac{\partial}{\partial v_{xx}} + \omega_1\frac{\partial}{\partial f_t} + \omega_2\frac{\partial}{\partial f_x} + \omega_3\frac{\partial}{\partial f_v} + \omega_4\frac{\partial}{\partial f_{v_t}} + ...,$$

where

$$\zeta_j\ ,\ \zeta_{jk} \qquad j,k = 1,2$$

are given in (2.3) and

$$\omega_i = \widetilde{D}_i(\mu) - f_t\widetilde{D}_i\left(\xi^1\right) - f_x\widetilde{D}_i\left(\xi^2\right) - f_v\widetilde{D}_i(\eta) - f_{v_t}\widetilde{D}_i(\zeta_1) - f_{v_x}\widetilde{D}_i(\zeta_2)\,,\ i = 1,...,4. \qquad (2.16)$$

The infinitesimal criterion for invariance of the system

$$v_{tt} = f v_{xx},$$

$$f_t = 0,\ f_x = 0,\ f_v = 0,\ f_{v_t} = 0, \tag{2.17}$$

which is written in the form

$$\widehat{E}\left[v_{tt} - f v_{xx}\right] |_{(1.1)} = 0. \tag{2.18}$$

Subject to equation (2.17) being satisfied we have

$$\widehat{E}\left(\ f_t\right) = \widehat{E}\left(\ f_x\right) = \widehat{E}\left(\ f_v\right) = \widehat{E}\left(\ f_{v_t}\right) = 0. \tag{2.19}$$

The equations (2.18) and (2.19) yield the following determining equations

$$\zeta_{11} - f\zeta_{22} - \mu v_{xx} = 0, \tag{2.20}$$

$$\omega_i = 0,\ i = 1, ..., 4. \tag{2.21}$$

Since

$$\omega_1 = \widetilde{D}_t\left(\mu\right) - f_t\widetilde{D}_t\left(\xi^1\right) - f_x\widetilde{D}_t\left(\xi^2\right) - f_v\widetilde{D}_t\left(\eta\right) - f_{v_t}\widetilde{D}_t\left(\zeta_1\right) - f_{v_x}\widetilde{D}_t\left(\zeta_2\right),$$

then together with (2.17) we have

$$\omega_1 = \left(\frac{\partial}{\partial t} + f_t\frac{\partial}{\partial f} + f_{tt}\frac{\partial}{\partial f_t} + f_{xt}\frac{\partial}{\partial f_x} + f_{tv}\frac{\partial}{\partial f_v} + f_{tv_t}\frac{\partial}{\partial f_{v_t}} + ...\right)\mu -$$

$$f_{v_x}\left(\frac{\partial}{\partial t} + f_t\frac{\partial}{\partial f} + f_{tt}\frac{\partial}{\partial f_t} + f_{xt}\frac{\partial}{\partial f_x} + f_{tv}\frac{\partial}{\partial f_v} + f_{tv_t}\frac{\partial}{\partial f_{v_t}} + ...\right)\zeta_2 = 0.$$

That is

$$\omega_1 = \frac{\partial \mu}{\partial t} - f_{v_x}\frac{\partial \zeta_2}{\partial t} = 0, \tag{2.22}$$

similarly

$$\omega_2 = \frac{\partial \mu}{\partial x} - f_{v_x}\frac{\partial \zeta_2}{\partial x} = 0, \tag{2.23}$$

$$\omega_3 = \frac{\partial \mu}{\partial v} - f_{v_x}\frac{\partial \zeta_2}{\partial v} = 0, \tag{2.24}$$

$$\omega_4 = \frac{\partial \mu}{\partial v_t} - f_{v_x}\frac{\partial \zeta_2}{\partial v_t} = 0. \tag{2.25}$$

Since f is a differential variable which is algebraically independent from f_{v_x}then the equations (2.22), (2.23), (2.24), (2,25) decompose with respect to f_{v_x}, hence we have

$$(\zeta_2)_x = (\zeta_2)_t = (\zeta_2)_v = (\zeta_2)_{v_t} = 0, \tag{2.26}$$

$$\mu_t = \mu_x = \mu_v = \mu_{v_t} = 0. \tag{2.27}$$

Since

$$\zeta_2 = \eta_x + v_x\eta_v - v_t\xi^1_x - v_tv_x\xi^1_v - v_x\xi^2_x - v_x^2\xi^2_v, \tag{2.28}$$

then we have

$$(\zeta_2)_x = \eta_{xx} + v_x\eta_{xv} - v_t\xi^1_{xx} - v_tv_x\xi^1_{vx} - v_x\xi^2_{xx} - v_x^2\xi^2_{vx} = 0. \tag{2.29}$$

Equation (2.29) splits into the following equations

$$\eta_{xx} = 0,\ \xi^1_{xx} - \xi^1_{xv}v_x = 0,\ \eta_{xv} - \xi^2_{xx} = 0,\ \xi^2_{xv} = 0. \tag{2.30}$$

Similarly

$$(\zeta_2)_t = \eta_{xt} + v_x\eta_{vt} - v_t\xi^1_{xt} - v_tv_x\xi^1_{vt} - v_x\xi^2_{xt} - v_x^2\xi^2_{vt} = 0, \tag{2.31}$$

splits into

$$\eta_{xt} = 0,\ \xi^1_{xt} - \xi^1_{tv}v_x = 0,\ \eta_{tv} - \xi^2_{xt} = 0,\ \xi^2_{tv} = 0 \tag{2.32}$$

and

$$(\zeta_2)_v = \eta_{xv} + v_x\eta_{vv} - v_t\xi^1_{xv} - v_tv_x\xi^1_{vv} - v_x\xi^2_{vx} - v_x^2\xi^2_{vv} = 0, \tag{2.33}$$

splits into

$$\eta_{xv} = 0,\ \eta_{vv} - \xi^2_{xv} = 0,\ \xi^1_{xv} - \xi^1_{vv} v_x = 0,\ \xi^2_{vv} = 0. \tag{2.34}$$

Finally

$$(\zeta_2)_{v_t} = \xi^1_x - v_x \xi^1_v = 0. \tag{2.35}$$

The equations (2.30), (2.32), (2.34) and (2.35) further split and are solved as follows

$$\xi^1_x = \xi^1_v = 0$$

and so

$$\xi^1 = \xi^1(t).$$

From

$$\xi^2_{vv} = \xi^2_{xx} = \xi^2_{xv} = \xi^2_{vt} = 0,$$

we have

$$\xi^2 = k_1 v + b_1(t) x + b_2(t).$$

Furthermore

$$\eta_{vv} = \eta_{vx} = \eta_{xx} = \eta_{xt} = 0,$$

implies

$$\eta = d_1(t) v + k_2 x + d_2(t),$$

where k_1 and k_2 are constants. From the invariance condition (2.18) we have the equation

$$\zeta_{11} - f\zeta_{22} - \mu v_{xx} \mid_{v_{tt} = f v_{xx}} = 0,$$

splitting into the following equations

$$\eta_{tt} = 0, \xi^2_t = 0, \xi^2_v = 0, \tag{2.36}$$

$$2\eta_{tv} - \xi^1_{tt} = 0, \tag{2.37}$$

$$f\left(2\xi^2_v v_x + 2\xi^2_x - 2\xi^1_t\right) = \mu. \tag{2.38}$$

These equations are solved as follows :

from

$$\xi^2 = k_1 v + b_1(t)\,x + b_2(t)$$

and

$$\xi^2_v = \xi^2_t = 0,$$

we have

$$\xi^2 = c_1 x + c_2.$$

Since μ and f are independent of the variable t, differentiating equation (2.38) with respect to t we obtain

$$\xi^1_{tt} = 0,$$

hence

$$\xi^1 = c_3 t + c_4.$$

Also

$$\eta = d_1(t)\,v + k_2 x + d_2(t)$$

and

$$\eta_{vt} = \eta_{tt} = 0$$

implies

$$\eta = c_5 v + c_6 x + c_7 t + c_8.$$

Finally

$$\mu = 2f(c_1 - c_3),$$

where $c_1, ..., c_8$ are constants. Thus we obtain an 8-dimensional equivalence algebra spanned by

$$\begin{aligned}
E_1 &= x\frac{\partial}{\partial x} + 2f\frac{\partial}{\partial f},\\
E_2 &= t\frac{\partial}{\partial t} - 2f\frac{\partial}{\partial f},\\
E_3 &= \frac{\partial}{\partial x},\\
E_4 &= \frac{\partial}{\partial t},\\
E_5 &= v\frac{\partial}{\partial v},\\
E_6 &= x\frac{\partial}{\partial v},\\
E_7 &= t\frac{\partial}{\partial v},\\
E_8 &= \frac{\partial}{\partial v}.
\end{aligned} \tag{2.39}$$

Solving the Lie equations for (2.39) we obtain the equivalence transformations

$$\begin{aligned}
\overline{x} &= \beta_1 x + \beta_3,\\
\overline{t} &= \beta_2 t + \beta_4,\\
\overline{v} &= \beta_5 v + \beta_6 x + \beta_7 t + \beta_8,\\
\overline{f} &= \beta_1^2 \beta_2^{-2} f,
\end{aligned} \tag{2.40}$$

where $\beta_1, ..., \beta_8$ are constants. We notice that the equation

$$f\left(-2\xi_t^1 + 2\xi_x^2\right) = f'\left[\eta_x + \left(\eta_v - \xi_x^2\right) v_x\right] \tag{2.41}$$

is equivalent to the relation

$$2f\,(a-b) = f'\,[c + (d-a)\,v_x] \tag{2.42}$$

with constant coefficients a, b, c and d. Since f depends only on v_x it is only possible for equation (2.41) to hold when all its coefficients vanish identically or are proportional to some function $\lambda(t,x,v) \neq 0$. We observe that if all the coefficients in (2.41) are simultaneously equal to zero, then this corresponds to the case of an arbitrary function f . The extension of the principal Lie algebra is only possible for functions f satisfying an equation of the form (2.42) with constant coefficients a, b, c and d such that $(a-b)$ and $c+(d-a)v_x$ are not zero. We obtain a classifying relation

$$\frac{f'}{f} = \frac{2\,(a-b)}{c + (d-a)\,v_x}. \tag{2.43}$$

2.3 Results of the classification

We shall now analyze the classifying relation (2.43). An equivalence relation of equations (1.1) can be carried out on (2.43). After equivalence transformations (2.40), equation (2.43) assumes the form

$$\frac{\overline{f'}}{\overline{f}} = \frac{2\left(\overline{a}-\overline{b}\right)}{\overline{c} + \left(\overline{d}-\overline{a}\right)\overline{v_{\overline{x}}}}. \tag{2.44}$$

We have

$$\overline{f'} = \frac{d\overline{f}}{d\overline{v_{\overline{x}}}} = \frac{D_{v_x}\left(\overline{f}\right)}{D_{v_x}\left(\overline{v_{\overline{x}}}\right)}, \tag{2.45}$$

Where

$$D_{v_x} = \frac{\partial}{\partial v_x} + f_{vx}\frac{\partial}{\partial f} + f_{v_x v_x}\frac{\partial}{\partial f_{v_x}} + f_{xv_x}\frac{\partial}{\partial f_x} + f_{tv_x}\frac{\partial}{\partial f_t} + f_{vv_x}\frac{\partial}{\partial f_v} + \ldots$$

and

$$\overline{v}_{\overline{x}} = \frac{D_x(\overline{v})}{D_x(\overline{x})} = \frac{\beta_6 + \beta_5 v_x}{\beta_1}. \tag{2.46}$$

We have

$$D_{v_x}(\overline{v}_{\overline{x}}) = \frac{\beta_5}{\beta_1}, \tag{2.47}$$

$$D_{v_x}\left(\overline{f}\right) = \beta_1^2\beta_2^{-2} f'. \tag{2.48}$$

Therefore

$$\overline{f}' = \frac{\beta_1^3\beta_2^{-2} f'}{\beta_5} \tag{2.49}$$

and

$$\overline{f} = \beta_1^2\beta_2^{-2} f. \tag{2.50}$$

Now

$$\frac{\overline{f}'}{\overline{f}} = \frac{\beta_1 f'}{\beta_5 f} = \frac{2\left(\overline{a} - \overline{b}\right)}{\overline{c} + \left(\overline{d} - \overline{a}\right)\left(\frac{\beta_6 + \beta_5 v_x}{\beta_1}\right)}, \tag{2.51}$$

implies

$$\frac{f'}{f} = \frac{2\left(\beta_5\overline{a} - \beta_5\overline{b}\right)}{\beta_1\overline{c} + \left(\overline{d} - \overline{a}\right)\beta_6 + \left(\overline{d} - \overline{a}\right)\beta_5 v_x}. \tag{2.52}$$

We observe that the coefficients $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{d}$ relate to the coefficients a, b, c and d by the formulae

$$a = \overline{a}\beta_5,\ b = \overline{b}\beta_5,\ d = \overline{d}\beta_5,\ c = \beta_1\overline{c} + \left(\overline{d} - \overline{a}\right)\beta_6. \tag{2.53}$$

We now use the above relation to obtain the non-equivalent forms of f . Three cases arise.

CASE 1 $c \neq 0$, $d - a = 0$.

Subcase 1.1 If $a \neq 0$, let $b = 0$.

The equation (2.43) has the form

$$\frac{f'}{f} = \frac{2a}{c},$$

hence

$$f = \alpha e^{\frac{2a}{c} v_x}, \qquad \alpha \in \Re.$$

Subcase 1.2 If $b \neq 0$, let $a = 0$.

The equation (2.43) has the form

$$\frac{f'}{f} = -\frac{2b}{c},$$

hence

$$f = \alpha e^{-\frac{2b}{c} v_x}, \qquad \alpha \in \Re.$$

Subcase 1.3 We let $a \neq 0$ and $b \neq 0$.

The equation (2.43) has the form

$$\frac{f'}{f} = \frac{2\,(a - b)}{c},$$

and so

$$f = \alpha e^{\frac{2(a-b)}{c} v_x}, \qquad \alpha \in \Re.$$

CASE 2 We let $d - a \neq 0$ and $c = 0$.

Subcase 2.1 If $a - b \neq 0$ and $a \neq 0$.

The equation (2.43) has the form

$$\frac{f'}{f} = \frac{2\,(a - b)}{(d - a)\, v_x},$$

thus

$$f = \alpha v_x^{\frac{2(a-b)}{d-a}}, \qquad \alpha \in \Re.$$

Subcase 2.2 If $a \neq 0$, let $d = 0$ and $b = 0$.

The equation (2.43) has the form

$$\frac{f'}{f} = -\frac{2}{v_x},$$

thus

$$f = \alpha v_x^{-2}, \qquad \alpha \in \Re.$$

Subcase 2.3 If $a \neq 0$, let $d \neq 0$ and $b = 0$.

The equation (2.43) has the form

$$\frac{f'}{f} = \frac{2a}{(d-a)\,v_x},$$

hence

$$f = \alpha v_x^{\frac{2a}{d-a}}, \qquad \alpha \in \Re.$$

Subcase 2.4 If $a \neq 0$, let $d = 0$ and $b \neq 0$.

The equation (2.43) has the form

$$\frac{f'}{f} = -\frac{2\,(a-b)}{a v_x},$$

hence

$$f = \alpha v_x^{-\frac{2(a-b)}{a}}, \qquad \alpha \in \Re.$$

CASE 3 Let $d - a \neq 0$ and $c \neq 0$.

The equation (2.43) has the form

$$\frac{f'}{f} = \frac{2\,(a-b)}{c + (d-a)\,v_x},$$

thus

$$f = [c + (d-a)\, v_x]^{\frac{2(a-b)}{d+a}}.$$

Substituting each of the f 's obtained above in (2.42) and then solving, we find those f 's for which the principal Lie algebra extends. For instance, in subcase 1.3 where $a \neq 0$, $b \neq 0$, $d - a = 0$ and $c \neq 0$,

we have

$$\frac{f'}{f} = \frac{2\,(a-b)}{c}.$$

If we let

$$\beta = \frac{2\,(a-b)}{c}, \qquad \beta \in \Re,$$

then

$$a = \frac{\beta c}{2} + b.$$

We have

$$\eta = cx + \left(\frac{\beta c}{2} + b\right) v + c_4 t + c_5,$$

$$\xi^2 = \left(\frac{\beta c}{2} + b\right) x + c_3,$$

$$\xi^1 = bt + c_2,$$

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hence

$$X_6 = \beta x \frac{\partial}{\partial x} + (2x + \beta v)\, \frac{\partial}{\partial v}.$$

For $\beta = 1$ we have

$$X_6 = x \frac{\partial}{\partial x} + (2x + v)\, \frac{\partial}{\partial v}. \tag{2.54}$$

In other words for $f = \alpha e^{v_x}$, the principal Lie algebra is extended by the generator (2.54).

Similarly in subcase 2.1, we let

$$\sigma = \frac{2(a-b)}{d-a},$$

then

$$a = \frac{\sigma d}{\sigma+2} + \frac{2b}{\sigma+2}, \quad \sigma \neq -2, 0.$$

That is for $f = \alpha v_x^\sigma$ we obtain the generator

$$X_6 = \sigma x \frac{\partial}{\partial x} + (\sigma+2)\, v \frac{\partial}{\partial v},$$

for $\sigma = -4$, that is $f = \alpha v_x^{-4}$, we have

$$X_6 = 2x \frac{\partial}{\partial x} + v \frac{\partial}{\partial v}$$

and for $\sigma = -\frac{4}{3}$, that is $f = \alpha v_x^{-\frac{4}{3}}$, we obtain

$$X_6 = 2x \frac{\partial}{\partial x} - v \frac{\partial}{\partial v}.$$

These results are given in [8], Volume 3, section 9:2. Some other method other than the equivalence is needed to find X_7 in the case of $f = \alpha v_x^{-4}$ and $f = \alpha v_x^{-\frac{4}{3}}$. Moreover quasilocal symmetries with the nonlocal variable w defined by the equation $w_x = v$, $w_{tt} = -3\delta v_x^{-\frac{1}{3}}$ are listed.

Chapter 3

Group classification of the equations $v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon g(v_x)$

Differential equations that models some phenomena in nature, often involve undetermined parameters and /or arbitrary functions of certain variables. In most cases these arbitrary functions or parameters are determined experimentally or chosen from a simple or trivial criteria. Lie group theory provides a regular procedure to determine these arbitrary functions or parameters from symmetry point of view. This study is commonly known as *Lie group classification of differential equations.* An essential part of the group classification is the utilization of equivalence transformations, which allow us to divide the set of all differential equations of the family into disjoint classes of equivalent equations. We shall start by introducing approximate symmetry groups and construct approximate equivalence transformations for the equations

$$v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon g(v_x). \tag{3.1}$$

3.1 The approximate symmetry groups

We wish to introduce the one-parameter approximate symmetry group for equations with small parameter. We consider the function $f(x, u, ..., \epsilon)$ of the form

$$f(x, u, ..., \epsilon) \equiv f_0(x, u, ...) + \epsilon f_1(x, u, ...) = 0, \tag{3.2}$$

where $f_0(x, u, ...) = 0$ is the unperturbed equation. For more details on this section see [9].

Consider the transformations of the variables (x, u) into $(\overline{x}, \overline{u})$.

$$\overline{x}^i = \varphi_0^i(x, u) + \epsilon\varphi_1^i(x, u), \ \overline{u}^k = \psi_0^k(x, u) + \epsilon\psi_1^k(x, u), \ i = 1, ..., n.k = 1, ..., m.$$

Definition 2 *The class of transformations of the form*

$$\overline{x}^i = \varphi^i(x, u, \epsilon), \overline{u}^k = \psi^k(x, u, \epsilon), i = 1, ..., n.k = 1, ..., m. \tag{3.3}$$

with functions

$$\varphi^i(x, u, \epsilon) \approx \varphi_0^i(x, u) + \epsilon\varphi_1^i(x, u); \ \psi^k(x, u, \epsilon) \approx \ \psi_0^k(x, u) + \ \epsilon\psi_1^k(x, u)$$

is called an approximate transformation.

Definition 3 *An approximate transformation of the variables (x, u) of the form (3.3) is said to be an approximate symmetry transformation of the equation (3.2) if it preserves the corresponding approximate equation up to order ϵ.*

Consider a one parameter family of invertible approximate transformations of the variables x, u given by

$$\overline{x}^i = \varphi^i(x, u, a, \epsilon), \overline{u}^k = \psi^k(x, u, a, \epsilon), i = 1, ..., n.k = 1, ..., m \tag{3.4}$$

such that

$$\varphi\mid_{a=0}\approx x^i,\ \psi\mid_{a=0}\approx u^k. \tag{3.5}$$

Definition 4 *The approximate transformations (3.4) satisfying the condition (3.5) form approximate one parameter transformation group G if*

$$\begin{aligned}\varphi^i(\overline{x},\overline{u},b,\epsilon) &= \varphi^i(x,u,a+b,\epsilon),\ \psi^k(\overline{x},\overline{u},b,\epsilon)=\psi^k(x,u,a+b,\epsilon),\\ i &= 1,...,n,\ k=1,...,m.\end{aligned}$$

If the transformations (3.4) of the group G are approximate symmetry transformations of the equation (3.2) then G is called the *approximate symmetry group* .

For approximate transformation groups, the operators are given by

$$X=\xi^i(x,u,\epsilon)\frac{\partial}{\partial x^i}+\eta^k(x,u,\epsilon)\frac{\partial}{\partial u^k},$$

where

$$\xi^i(x,u,\epsilon)\approx\xi_0^i(x,u)+\epsilon\xi_1^i(x,u)$$

$$\eta^k(x,u,\epsilon)\approx\eta_0^k(x,u)+\epsilon\eta_1^k(x,u),$$

hence

$$X=\left(\xi_0^i(x,u)+\epsilon\xi_1^i(x,u)\right)\frac{\partial}{\partial x^i}+\left(\eta_0^k(x,u)+\epsilon\eta_1^k(x,u)\right)\frac{\partial}{\partial u^k}.$$

One can write

$$X=X_0+\epsilon X_1. \tag{3.6}$$

The operator X_0 is called the *stable symmetry* if it is admitted by the unperturbed equation. The corresponding approximate symmetry generator X for the perturbed equation is called a *deformation* of the operator X_0.

3.2 The approximate equivalence transformations

We wish to calculate the approximate equivalence transformations for the equations

$$v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon g\left(v_x\right), \ \alpha = \pm 1. \tag{3.7}$$

In this case, a natural modification of equivalence transformation that involves approximate transformations is used.

Definition 5 *An approximate equivalence transformation is a nondegerate (at $\varepsilon = 0$) change of variables of the form*

$$\begin{aligned} \overline{x} &= \varphi_0^1(x,t,v) + \epsilon\varphi_1^1(x,t,v), \ \overline{t} = \varphi_0^2(x,t,v) + \epsilon\varphi_1^2(x,t,v) \\ \overline{v} &= \psi_0(x,t,v) + \epsilon\psi_1(x,t,v), \end{aligned}$$

such that, in the precision indicated, the equation (1.3) is written as

$$\overline{v}_{\overline{tt}} = \alpha e^{\overline{v}_{\overline{x}}} \overline{v}_{\overline{xx}} + \epsilon\overline{g}\left(\overline{v}_{\overline{x}}\right) + o(\epsilon).$$

That is the form of the equation (1.3) is not changed.

The algorithm for finding the approximate equivalence transformations is similar to that used in the case of exact equivalence transformation groups [5]. We consider the system of equations

$$\begin{aligned} v_{tt} - \alpha e^{v_x} v_{xx} - \epsilon g\left(v_x\right) &= o(\epsilon) \\ \epsilon g_x = \epsilon g_t = \epsilon g_v = \epsilon g_{v_t} &= o(\epsilon) \end{aligned} \tag{3.8}$$

and seek the operator

$$E = E_0 + \epsilon E_1 = \left(\xi^0 + \epsilon\xi^1\right)\frac{\partial}{\partial x} + \left(\tau^0 + \epsilon\tau^1\right)\frac{\partial}{\partial t} + \left(\eta^0 + \epsilon\eta^1\right)\frac{\partial}{\partial v} + \mu\frac{\partial}{\partial g}.$$

where $\xi^{\nu}, \tau^{\nu}, \eta^{\nu}$ $(\nu = 0, 1)$ are functions of t, x and v, but μ depends on the variables t, x, v, v_x, v_t and g. In the extended space we have

$$\begin{aligned}\overline{E} &= E + \left(\zeta_0^x + \epsilon\zeta_1^x\right)\frac{\partial}{\partial v_x} + \left(\zeta_0^t + \epsilon\zeta_1^t\right)\frac{\partial}{\partial v_t} + \left(\zeta_0^{xx} + \epsilon\zeta_1^{xx}\right)\frac{\partial}{\partial v_{xx}} + \\ &\left(\zeta_0^{tt} + \epsilon\zeta_1^{tt}\right)\frac{\partial}{\partial v_{tt}} + \omega_v\frac{\partial}{\partial g_v} + \omega_t\frac{\partial}{\partial g_t} + \omega_x\frac{\partial}{\partial g_x} + \omega_{v_t}\frac{\partial}{\partial g_{v_t}} + \ldots \qquad (3.9)\end{aligned}$$

where

$$\begin{aligned}\omega_a &= \widetilde{D}_a(\mu) - g_t\widetilde{D}_a\left(\tau^0 + \epsilon\tau^1\right) - g_x\widetilde{D}_a\left(\xi^0 + \epsilon\xi^1\right) - g_v\widetilde{D}_a\left(\eta^0 + \epsilon\eta^1\right) \\ &-g_{v_t}\widetilde{D}_a\left(\zeta_0^t + \epsilon\zeta_1^t\right) - g_{v_x}\widetilde{D}_a\left(\zeta_0^x + \epsilon\zeta_1^x\right) \qquad (3.10)\end{aligned}$$

and

$$\begin{aligned}\widetilde{D}_a &= \frac{\partial}{\partial a} + g_a\frac{\partial}{\partial g} + g_{at}\frac{\partial}{\partial g_t} + g_{ax}\frac{\partial}{\partial g_x} + g_{av}\frac{\partial}{\partial g_v} + g_{av_t}\frac{\partial}{\partial g_{v_t}} + \ldots, \\ a &\in \{x, t, v, v_t\}.\end{aligned}$$

The formulae for the functions $\zeta_0^x, \zeta_1^x, \zeta_0^t, \zeta_1^t, \zeta_0^{xx}, \zeta_1^{xx}, \zeta_0^{tt}$ and ζ_1^{tt} are given in appendix A. Since

$$g_a = 0, \ \forall\ a \in \{x, t, v, v_t\},$$

then

$$\widetilde{D}_a = \frac{\partial}{\partial a}.$$

The infinitesimal approximate invariance criterion for the system (3.8) is written as

$$\overline{E}\left(v_{tt} - \alpha e^{v_x}v_{xx} - \epsilon g\left(v_x\right)\right)|_{(3.7)} = o\left(\epsilon\right) \qquad (3.11)$$

and subject to the satisfaction of equation (3.8) we have

$$\overline{E}\left(\epsilon g_x\right) = \overline{E}\left(\epsilon g_t\right) = \overline{E}\left(\epsilon g_v\right) = \overline{E}\left(\epsilon g_{v_t}\right) = o\left(\epsilon\right). \qquad (3.12)$$

3.2.1 The zero order terms

In the zero-order of precision, equation (3.11) becomes

$$\overline{E}\left(v_{tt}-\alpha e^{v_x}v_{xx}\right)|_{v_{tt}=\alpha e^{v_x}v_{xx}}=0. \tag{3.13}$$

Together with (3.12) we obtain

$$\zeta_0^{tt}-\alpha\zeta_0^{x}e^{v_x}v_{xx}-\alpha\zeta_0^{xx}e^{v_x}=0, \tag{3.14}$$

$$\epsilon\omega_v=\epsilon\omega_t=\epsilon\omega_x=\epsilon\omega_{v_t}=0. \tag{3.15}$$

From equations (3.15) we have

$$\mu_a-g_{v_x}\left(\zeta_0^x\right)_a=0, \qquad a\in\{x,t,v,v_t\}. \tag{3.16}$$

Since g is a differential variable which is algebraically independent from g_{v_x} then the equations (3.16) decompose with respect to g_{v_x}, hence we have

$$\mu_a=0 \quad \text{and} \quad \left(\zeta_0^x\right)_a=0,\ \forall \quad a\in\{x,t,v,v_t\}. \tag{3.17}$$

Thus μ is a function of v_x and g only. Since

$$\zeta_0^x=\eta_x^0+v_x\eta_v^0-v_t\tau_x^0-v_tv_x\tau_v^0-v_x\xi_x^0-v_x^2\xi_v^0, \tag{3.18}$$

then we have

$$\left(\zeta_0^x\right)_x=\eta_{xx}^0+v_x\eta_{xv}^0-v_t\tau_{xx}^0-v_tv_x\tau_{vx}^0-v_x\xi_{xx}^0-v_x^2\xi_{vx}^0=0. \tag{3.19}$$

Equation (3.19) splits into the following equations

$$\eta_{xx}^0=0,\ \tau_{xx}^0=0,\ \tau_{xv}^0=0,\ \eta_{xv}^0-\xi_{xx}^0=0,\ \xi_{xv}^0=0. \tag{3.20}$$

Similarly

$$(\zeta_0^x)_t = \eta_{xt}^0 + v_x\eta_{vt}^0 - v_t\tau_{xt}^0 - v_tv_x\tau_{vt}^0 - v_x\xi_{xt}^0 - v_x^2\xi_{vt}^0 = 0, \tag{3.21}$$

splits into

$$\eta_{xt} = 0,\ \tau_{xt}^0 = 0,\ \tau_{tv}^0 = 0,\ \eta_{tv}^0 - \ \xi_{xt}^0 = 0,\ \xi_{tv}^0 = 0, \tag{3.22}$$

also

$$(\zeta_0^x)_v = \eta_{xv}^0 + v_x\eta_{vv}^0 - v_t\tau_{xv}^0 - v_tv_x\tau_{vv}^0 - v_x\xi_{vx}^0 - v_x^2\xi_{vv}^0 = 0, \tag{3.23}$$

splits into

$$\eta_{xv}^0 = 0,\ \eta_{vv}^0 - \xi_{xv}^0 = 0,\ \tau_{xv}^0 = 0,\ \tau_{vv}^0 = 0,\ \xi_{vv}^0 = 0. \tag{3.24}$$

Finally

$$(\zeta_2)_{v_t} = \tau_x^0 - v_x\tau_v^0 = 0,$$

splits into

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$$\tau_x^0 = 0,\ \tau_v^0 = 0. \tag{3.25}$$

From equations (3.20), (3.22), (3.24) and (3.25) we have

$$\xi^0 = a_1v + e_1(t)x + e_2(t),$$

$$\tau^0 = \tau^0(t), \tag{3.26}$$

$$\eta^0 = b_1(t)v + a_2x + b_2(t),$$

where a_1 and a_2 are constant coefficients. Equations (3.14) decompose into the equations

$$\xi_v^0 = \xi_t^0 = \tau_v^0 = \tau_x^0 = \eta_{tt}^0 = 0,$$

$$\eta_{xt}^0 = \eta_{tv}^0 = \eta_{xv}^0 = \eta_{vv}^0 = \eta_{xx}^0 = 0,$$

$$2\xi_x^0 - 2\tau_t^0 - \eta_x^0 = 0,$$

$$\eta_v^0 - \xi_x^0 = 0,$$

$$\xi_{xx}^0 = \tau_{tt}^0 = 0. \tag{3.27}$$

From equations (3.27) we obtain

$$\xi^0 = (c_5 + c_6)\, x + c_2,$$

$$\tau^0 = c_5 t + c_1, \tag{3.28}$$

$$\eta^0 = 2c_6 x + (c_5 + c_6)\, v + c_4 t + c_3,$$

with constant coefficients $c_1, ..., c_6$. Thus we obtain 6 stable symmetries given by

$$X_1 = \frac{\partial}{\partial t},$$

$$X_2 = \frac{\partial}{\partial x},$$

$$X_3 = \frac{\partial}{\partial v},$$

$$X_4 = t\frac{\partial}{\partial v},$$

$$X_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v}, \tag{3.29}$$

$$X_6 = x\frac{\partial}{\partial x} + (v + 2x)\,\frac{\partial}{\partial v}.$$

3.2.2 The first order terms

In the first-order of precision, the invariance condition on the equation (3.7)

$$\overline{E}\left(v_{tt} - \alpha e^{v_x} v_{xx} - \epsilon g\left(v_x\right)\right) |_{(3.7)} = o\left(\epsilon\right), \tag{3.30}$$

yields the determining equation

$$\left(\zeta_0^{tt}+\epsilon\zeta_1^{tt}\right)-\alpha\left(\zeta_0^{x}+\epsilon\zeta_1^{x}\right)e^{v_x}v_{xx}-\alpha\left(\zeta_0^{xx}+\epsilon\zeta_1^{xx}\right)e^{v_x}-\epsilon\mu=0.$$

Substituting

$$v_{tt}=\alpha e^{v_x}v_{xx}+\epsilon g\left(v_x\right),$$

we obtain the determining equation

$$\left(\eta_v^0-2\tau_t^0-3\tau_v^0-\xi_v^0 v_x\right)g+\zeta_1^{tt}-\alpha e^{v_x}v_{xx}\zeta_1^{x}-\alpha e^{v_x}\zeta_1^{xx}-\mu=0. \tag{3.31}$$

Equations (3.31) split into the equations

$$\begin{gathered}
\tau_v^1=\tau_x^1=\xi_v^1=\xi_t^1=0,\\
\eta_v^1-\xi_x^1=0,\\
2\xi_x^1-2\tau_t^1-\eta_x^1=0,\\
\eta_{vv}^1=0,\\
2\eta_{tv}^1-\tau_{tt}^1=0,\\
\left(\eta_v^0-2\tau_t^0\right)g+\eta_{tt}^1-\xi_{tt}^1 v_x-\alpha e^{v_x}\{\eta_{xx}^1+\left(2\eta_{xv}^1-\xi_{xx}^1\right)v_x\}=\mu.
\end{gathered} \tag{3.32}$$

Solving the equations (3.32) we obtain

$$\begin{gathered}
\tau^1=a_1 t+a_2,\\
\xi^1=a_3 x+a_4,\\
\eta^1=a_3 v+2\left(a_1-a_3\right)x+\frac{a_5 t^2}{2}+a_6 t+a_7,\\
\mu=\left(c_1-2c_3\right)g+a_5,
\end{gathered} \tag{3.33}$$

with constant coefficients $a_1, ..., a_7$. Thus we obtain a 13-dimensional approximate equivalence Lie algebra spanned by the generators

$$
\begin{aligned}
E_1 &= \frac{\partial}{\partial t}, \\
E_2 &= \frac{\partial}{\partial x}, \\
E_3 &= \frac{\partial}{\partial v}, \\
E_4 &= t\frac{\partial}{\partial v}, \\
E_5 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - g\frac{\partial}{\partial g}, \\
E_6 &= x\frac{\partial}{\partial x} + (v+2x)\frac{\partial}{\partial v} + g\frac{\partial}{\partial g}, \\
E_7 &= \epsilon\frac{\partial}{\partial t}, \\
E_8 &= \epsilon\frac{\partial}{\partial x}, \\
E_9 &= \epsilon\frac{\partial}{\partial v}, \\
E_{10} &= \epsilon t\frac{\partial}{\partial v}, \\
E_{11} &= \epsilon\left(t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial v}\right), \\
E_{12} &= \epsilon\left(x\frac{\partial}{\partial x} + (v+2x)\frac{\partial}{\partial v}\right), \\
E_{13} &= \epsilon\left(t^2\frac{\partial}{\partial v} + 2\frac{\partial}{\partial g}\right).
\end{aligned}
\tag{3.34}
$$

Equation (3.7) admits a 13-dimensional Lie algebra of infinitesimal generators of a 13-parameter group of approximate equivalence transformations. The three nontrivial generators are

$$
E_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + v\frac{\partial}{\partial v} - g\frac{\partial}{\partial g},
$$

$$E_6 = x\frac{\partial}{\partial x} + (v+2x)\frac{\partial}{\partial v} + g\frac{\partial}{\partial g}, \tag{3.35}$$
$$E_{13} = \epsilon\left(t^2\frac{\partial}{\partial v} + 2\frac{\partial}{\partial g}\right).$$

It is sufficient, for group classification, to consider the point approximate equivalence transformations corresponding to (3.35). These transformations are given by

$$\begin{aligned}\overline{x} &= a_1a_2x, \quad \overline{t} = a_1t, \quad \overline{g} = \epsilon\left(2a_1a_2a_3 + a_1a_2g\right),\\ \overline{v} &= \epsilon\left[a_1a_2a_3t^2 - 2a_1\left(a_2-1\right)x + a_1a_2v\right]\end{aligned} \tag{3.36}$$

3.3 Results of the classification

In this section we wish to find the principal Lie algebra for the equations (3.7), furthermore we find those functions g for which the principal Lie algebra is extended. We seek the admitted operator in the form

$$\begin{aligned}X &= \left(\xi^0(t,x,v) + \epsilon\xi^1(t,x,v)\right)\frac{\partial}{\partial x} + \left(\tau^0(t,x,v) + \epsilon\tau^1(t,x,v)\right)\frac{\partial}{\partial t}\\ &\quad + \left(\eta^0(t,x,v) + \epsilon\eta^1(t,x,v)\right)\frac{\partial}{\partial v}.\end{aligned}$$

The prolonged operator is given by

$$\begin{aligned}\widetilde{X} &= X + \left(\zeta_0^x + \epsilon\zeta_1^x\right)\frac{\partial}{\partial v_x} + \left(\zeta_0^t + \epsilon\zeta_1^t\right)\frac{\partial}{\partial v_t} + \left(\zeta_0^{xx} + \epsilon\zeta_1^{xx}\right)\frac{\partial}{\partial v_{xx}} +\\ &\quad \left(\zeta_0^{tt} + \epsilon\zeta_1^{tt}\right)\frac{\partial}{\partial v_{tt}} + \ldots\end{aligned}$$

The invariance condition on equation (3.7)

$$\widetilde{X}\left(v_{tt} - \alpha e^{v_x}v_{xx} - \epsilon g\left(v_x\right)\right)|_{(3.7)} = o\left(\epsilon\right),$$

yields the determining equation

$$\zeta_0^{tt} + \epsilon\zeta_1^{tt} - \alpha\left(\zeta_0^x + \epsilon\zeta_1^x\right)e^{v_x}v_{xx} - \alpha\left(\zeta_0^{xx} + \epsilon\zeta_1^{xx}\right)e^{v_x} - \epsilon g'\left(\zeta_0^x + \epsilon\zeta_1^x\right) = 0.$$

In the zero order of precision we obtain similar results as in (3.28) whereas in the first order of precision we have the determining equation

$$\left(\eta_v^0 - 2\tau_t^0\right) g + \zeta_1^{tt} - \alpha e^{v_x} v_{xx} \zeta_1^x - \alpha e^{v_x} \zeta_1^{xx} - g' \zeta_x^0 = 0. \tag{3.37}$$

Substituting $v_{tt} = \alpha e^{v_x} v_{xx}$ and considering arbitrary g, the equations (3.37) split into

$$\begin{aligned}
&\xi_v^1 = 0,\ \tau_v^1 = 0,\ \tau_x^1 = 0,\ \xi_t^1 = 0,\ \eta_{tt}^1 = 0,\\
&\eta_x^1 = -2\left(\tau_t^1 - \xi_t^1\right), \quad \eta_{xx}^1 = 0,\\
&\eta_v^1 - \xi_x^1 = 0, \quad 2\eta_{tv}^1 - \tau_{tt}^1 = 0.\\
&2\eta_{xv}^1 - \xi_{xx}^1 = 0,\ \eta_{vv}^1 - 2\xi_{xv}^1 = 0,\\
&\eta_x^0 = 0,\ \eta_v^0 - \xi_x^0 = 0,\ \xi_v^0 = 0,\\
&\eta_v^0 - 2\tau_t^0 = 0.
\end{aligned} \tag{3.38}$$

Since

$$\eta_v^0 - 2\tau_t^0 = 0,$$

we then have that $c_5 = 0$. Thus we obtain

$$\xi^0 = c_2,$$

$$\tau^0 = c_1,$$

$$\eta^0 = c_4 t + c_3,$$

where $c_1, ..., c_4$ are constants, also from equation (3.37) we obtain

$$\xi^1 = a_1 x + a_2,$$

$$\tau^1 = a_3 t + a_4,$$

$$\eta^1 = a_1 v + 2\left(a_1 - a_3\right) x + a_5 t + a_6,$$

where $a_1, ..., a_6$ are constants. Thus the principal Lie algebra[1] is 10-dimensional and its basis is

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial v}, \\
X_2 &= \frac{\partial}{\partial x}, \\
X_3 &= \frac{\partial}{\partial t}, \\
X_4 &= t\frac{\partial}{\partial v}, \\
X_5 &= \epsilon\frac{\partial}{\partial v}, \\
X_6 &= \epsilon\frac{\partial}{\partial x} \\
X_7 &= \epsilon\frac{\partial}{\partial t}, \\
X_8 &= \epsilon t\frac{\partial}{\partial v}, \\
X_9 &= \epsilon\left(t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial v}\right), \\
X_{10} &= \epsilon\left(x\frac{\partial}{\partial x} + (2x+v)\frac{\partial}{\partial v}\right).
\end{aligned} \tag{3.39}$$

If we consider the function g not arbitrary, then equation (3.37) reduces to

$$\left(\eta^0_v - 2\tau^0_t\right) g + \eta^1_{tt} - g'\eta^0_x = 0. \tag{3.40}$$

The equation (3.40) is equivalent to the relation

$$(c_6 - c_5)g + \gamma - 2c_6 g' = 0. \tag{3.41}$$

where c_5, c_6 and γ are constant coefficients.

[1] We prove that these symmetries are admitted by equations (3.7) in appendix B

3.3.1 Analysis of the classifying relation

We use the relation (3.41) to obtain non-equivalent forms of g. Two cases arise.

CASE 1 If $\gamma = 0$ then (3.41) becomes

$$\frac{g'}{g} = \frac{c_6 - c_5}{2c_6},$$

thus

$$g = Ae^{\frac{c_6 - c_5}{2c_6} v_x},$$

where A is a constant. Let

$$\beta = \frac{c_6 - c_5}{2c_6}, \qquad \beta \neq 0,$$

then

$$(1 - 2\beta)\, c_6 = c_5,$$

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hence

$$\begin{aligned}
\tau^0 &= (1 - 2\beta)\, c_6 t + c_1, \\
\xi^C &= 2\,(1 - \beta)\, c_6 x + c_2, \\
\eta^0 &= 2c_6 x + 2\,(1 - \beta)\, c_6 v + c_4 t + c_3, \\
\xi^1 &= a_1 x + a_2, \\
\tau^1 &= a_3 t + a_4, \\
\eta^1 &= a_1 v + 2\,(a_1 - a_3) + a_5 t + a_6.
\end{aligned} \tag{3.42}$$

So we obtain the eleventh symmetry, namely

$$X_{11} = 2\,(1 - \beta)\, x \frac{\partial}{\partial x} + (1 - 2\beta)\, t \frac{\partial}{\partial t} + [2x + 2\,(1 - \beta)\, v]\, \frac{\partial}{\partial v}, \forall \beta \in \Re. \tag{3.43}$$

In other words the equation

$$v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon A e^{\beta v_x}, \qquad A > 0, \beta \in \Re, \alpha = \pm 1.$$

admits 11-dimensional Lie algebra.

In particular for $\beta = 1$, the generator (3.43) takes the form

$$X_{11} = -t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial v}$$

and for $\beta = \frac{1}{2}$ we obtain

$$X_{11} = x\frac{\partial}{\partial x} + (2x + v)\frac{\partial}{\partial v}.$$

CASE 2 If $\gamma \neq 0$ then (3.41) becomes

$$g' - \frac{c_6 - c_5}{2c_6} g = \frac{\gamma}{2c_6},$$

thus

$$g = \frac{-\gamma}{c_6 - c_5} + B e^{\frac{c_6 - c_5}{2c_6} v_x},$$

where B is a constant. Let

$$\delta = \frac{-\gamma}{c_6 - c_5}, \qquad \delta \neq 0,$$

and

$$\beta = \frac{c_6 - c_5}{2c_6}, \qquad \beta \neq 0,$$

then

$$\gamma = -\delta\,(c_6 - c_5) \qquad \text{and} \qquad (1 - 2\beta)\, c_6 = c_5.$$

Thus

$$\eta^0 = 2c_6 x + 2c_6\,(1 - \beta)\, v - 2\delta\beta c_6 t^2 + c_4 t + c_3,$$

$$\xi^0 = 2(1-\beta)\, c_6 x + c_2,$$

$$\tau^0 = (1-2\beta)\, c_6 t + c_1.$$

Thus we obtain the eleventh generator

$$X_{11} = 2(1-\beta)\, x\frac{\partial}{\partial x} + (1-2\beta)\, t\frac{\partial}{\partial t} + \left[2x + 2(1-\beta)\, v - 2\delta\beta t^2\right] \frac{\partial}{\partial v},$$

which is admitted by the equation

$$v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon \left(\delta + B e^{\beta v_x}\right).$$

We observe that the principal Lie algebra does not extend if g is a constant.

Chapter 4

The adjoint group and Invariant solutions

4.1 The adjoint group for the algebra L_{10}

In this section we shall construct the adjoint group of L_{10}. We start by giving some definitions and explanation of some terms. See Vol.2 in [8] for more details.

Definition 6 Lie algebra
A Lie algebra is a vector space L, such that for $X_1, X_2, X_3 \in L$, the bilinear product $[X_1, X_2]$, called commutator of X_1 and X_2, is an element in L. Moreover

$$[X_1, X_2] = -[X_2, X_1]$$

and the Jacobi identity

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$$

is satisfied.

We consider vector spaces over the field of real numbers. The dimension of the Lie algebra L is the dimension of the vector space. We shall denote an *r-dimensional* Lie algebra by L_r.

Definition 7 The structure constants
Let $X_1, X_2, ..., X_r$ be the basis of the vector space L_r. Then L_r is closed under the commutator if

$$[X_\mu, X_\nu] = c^\lambda_{\mu\nu} X_\lambda$$

where constant coefficients $c^\lambda_{\mu\nu}$ are known as the structure constants.

Definition 8 Isomorphism and Automorphism

Let L and K be two algebras that are isomorphic. The linear one-to-one and onto map

$$. f : L \rightarrow K$$

is said to be an isomorphism if

$$f\left([X_1, X_2]_L\right) = \left[f\left(X_1\right), f\left(X_2\right)\right]_K,$$

where the indexes L and K are used to denote the commutator in the corresponding algebra.

Two algebras are isomorphic if they have the same structure constants in an appropriately chosen basis. An isomorphism of L onto itself is called an automorphism.

Definition 9 Inner automorphism

Let $X_1, X_2, ..., X_r$ be the selected basis of the vector space L_r. Accordingly, the structure constants $c^{\lambda}_{\mu\nu}$ are known and any $X \in L$ is written as

$$X = e^{\mu} X_{\mu}.$$

Hence, the elements of L_r are represented by vectors $e = \left(e^1, ..., e^r\right)$. Let L^A_r be a Lie algebra spanned by the following operators

$$E_{\mu} = c^{\lambda}_{\mu\nu} e^{\nu} \frac{\partial}{\partial e^{\lambda}}, \qquad \mu = 1, ..., r.$$

with the commutator defined by the formula

$$[X_1, X_2] = X_1, X_2 - X_2 X_1.$$

The algebra L^A_r generates the group G^A of linear transformations of $\{e^{\mu}\}$. These transformations determine the automorphisms of the algebra L_r known as inner automorphisms. The group G^A is called group of automorphisms of L_r, or the adjoint group of L_r.

We now consider the commutators of L_{10} given in the table below

$[X_i,X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-\epsilon(2X_6-X_2)$	0	0	ϵX_5
X_2	$\epsilon(2X_6-X_2)$	0	$2\epsilon X_6$	0	0
X_3	0	$-2\epsilon X_6$	0	ϵX_4	0
X_4	0	0	$-\epsilon X_4$	0	$-\epsilon X_6$
X_5	$-\epsilon X_5$	0	0	ϵX_6	0
X_6	$-\epsilon X_6$	0	0	0	0
X_7	$2X_6-X_2$	0	$2X_6$	0	0
X_8	0	0	$-X_4$	0	$-X_6$
X_9	$-X_5$	0	X_5	X_6	0
X_{10}	$-X_6$	0	0	0	0

$[X_i,X_j]$	X_6	X_7	X_8	X_9	X_{10}
X_1	ϵX_6	$-(2X_6-X_2)$	0	X_5	X_6
X_2	0	0	0	0	0
X_3	0	$-2X_6$	X_4	$-X_5$	0
X_4	0	0	0	$-X_6$	0
X_5	0	0	X_6	0	0
X_6	0	0	0	0	0
X_7	0	0	0	0	0
X_8	0	0	0	$-X_{10}$	0
X_9	0	0	X_{10}	0	0
X_{10}	0	0	0	0	0

Table 4.1 : Commutators of L_{10}

We wish to determine the transformations that give rise to the adjoint group of L_{10}. The generators of the adjoint algebra L_{10}^A are in the form

$$E_\mu = c_{\mu\nu}^\lambda e^\nu \frac{\partial}{\partial e^\lambda}, \qquad \mu = 1, ..., 10. \tag{4.1}$$

and

$$[X_\mu, X_\nu] = c_{\mu\nu}^\lambda X_\lambda$$

We wish to determine the generator E_1 as an example and the rest follow in a similar manner.

Let

$$\mu = 1 \quad \text{and} \quad \lambda, \nu = 1, ..., 10.$$

We write the bracket as

$$[X_1, X_\nu] = c_{1\nu}^\lambda X_\lambda,$$

For $\nu = 2$ we have

$$[X_1, X_2] = c_{12}^\lambda X_\lambda = c_{12}^1 X_1 + c_{12}^2 X_2 + ... + c_{12}^{10} X_{10}$$

and so we obtain

$$c_{12}^2 = \epsilon \quad \text{and} \quad c_{12}^6 = -2\epsilon.$$

For $\nu = 5$ we obtain

$$c_{15}^5 = \epsilon,$$

for $\nu = 6$ we obtain

$$c_{16}^6 = \epsilon,$$

for $\nu = 7$ we obtain

$$c_{17}^6 = -2 \quad \text{and} \quad c_{17}^2 = 1,$$

for $\nu = 9$ we obtain

$$c_{19}^{5} = 1$$

and finally for $\nu = 10$ we have

$$c_{110}^{6} = 1.$$

The generator (4.1) has the form

$$E_1 = \left(\epsilon e^2 + e^7\right)\frac{\partial}{\partial e^2} + \left(\epsilon e^5 + e^9\right)\frac{\partial}{\partial e^5} + \left(\epsilon e^6 - 2e^7 - 2\epsilon e^2 + e^{10}\right)\frac{\partial}{\partial e^6}$$

and similarly

$$\begin{aligned}
E_2 &= 2\epsilon e^1\frac{\partial}{\partial e^6} - \epsilon e^1\frac{\partial}{\partial e^2} + 2\epsilon e^3\frac{\partial}{\partial e^6},\\
E_3 &= -\epsilon e^2\frac{\partial}{\partial e^6} + \epsilon e^4\frac{\partial}{\partial e^4} - 2e^7\frac{\partial}{\partial e^6} + e^8\frac{\partial}{\partial e^4} - e^9\frac{\partial}{\partial e^5},\\
E_4 &= \epsilon e^3\frac{\partial}{\partial e^4} - \epsilon e^5\frac{\partial}{\partial e^6} - e^9\frac{\partial}{\partial e^6},\\
E_5 &= -\epsilon e^1\frac{\partial}{\partial e^5} + \epsilon e^4\frac{\partial}{\partial e^6} + e^8\frac{\partial}{\partial e^6},\\
E_6 &= -\epsilon e^1\frac{\partial}{\partial e^6},\\
E_7 &= 2e^1\frac{\partial}{\partial e^6} - e^1\frac{\partial}{\partial e^2} + 2e^3\frac{\partial}{\partial e^6},\\
E_8 &= -e^3\frac{\partial}{\partial e^4} - e^5\frac{\partial}{\partial e^6} + e^9\frac{\partial}{\partial e^{10}},\\
E_9 &= -e^1\frac{\partial}{\partial e^5} + e^3\frac{\partial}{\partial e^5} + e^4\frac{\partial}{\partial e^6} + e^8\frac{\partial}{\partial e^{10}},\\
E_{10} &= -e^1\frac{\partial}{\partial e^6}.
\end{aligned} \tag{4.2}$$

From the operators (4.2) we solve the Lie equations to obtain the following adjoint transformations

$$\bar{e}^1 = e^1,$$

$$\overline{e}^2 = 2\left(a_2 - a_1^{\epsilon} a_3 a_7\right) e^1 - 2\epsilon a_1^{\epsilon} a_3 e^2 + \left(\frac{a_1^{\epsilon}}{\epsilon} - 1\right) e^7,$$

$$\overline{e}^3 = e^3,$$

$$\overline{e}^4 = (\epsilon + 1) a_3 e^4 - \{(\epsilon + 1) a_3\} \{a_8 + a_4 \epsilon\} e^3,$$

$$\overline{e}^5 = -\epsilon a_5 e^1 + a_1^{\epsilon} e^5 + \left(\frac{a_1^{\epsilon}}{\epsilon} - 1\right) e^9, \tag{4.3}$$

$$\begin{aligned} \overline{e}^6 \;&=\; a_1^{\epsilon} \left\{2\epsilon \left(e^1 + e^3\right) a_2 - 2a_3 \left(\epsilon e^2 + e^7\right) + a_4 \left(\epsilon e^5 + e^{10}\right)\right\} + a_5 \left(\epsilon e^4 + e^8\right) \\ &\quad -\epsilon a_6 e^1 + 2a_7 \left(e^1 + e^3\right) - a_8 e^5 + a_9 e^4 - a_{10} e^1 - 2e^7 + e^{10} + e^6, \end{aligned}$$

$$\overline{e}^7 = e^7,$$

$$\overline{e}^8 = e^8,$$

$$\overline{e}^9 = e^9,$$

$$\overline{e}^{10} = e^8 a_9 - e^9 a_8 + e^{10}.$$

These transformations give rise to the adjoint group elements of the algebra L_{10}

4.2 Some approximate invariant solutions

In this section we wish to construct some regular invariant approximate solutions for the equation (1.4). The algorithm for constructing the approximate invariant solution of differential equations with small perturbation can be found in [3], [9].

The equation (3.2) is said to be approximately invariant under the approximate group G if and only if

$$Xf \mid_{(3.2)} = o(\epsilon), \tag{4.4}$$

where the generator (3.6) of the group G is extended to the necessary derivatives. We say that the equation f admits the approximate generator X if (4.4) holds.

The approximate invariants for the operator (3.6) given by

$$J(t,x,v,\epsilon) = J_0(t,x,v) + \epsilon J_1(t,x,v)$$

are determined by the equation

$$XJ = o(\epsilon)$$

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or equivalently

$$X_0(J_0 + \epsilon J_1) + \epsilon X_1(J_0 + \epsilon J_1) \approx X_0 J_0 + \epsilon (X_0 J_1 + X_1 J_0) \approx 0. \tag{4.5}$$

Equation (4.5) splits into two equations

$$X_0 J_0 \approx 0 \quad \text{and} \quad X_0 J_1 \approx -X_1 J_0.$$

Among other generators the equation (1.4) admits the generators

$$Y_1 = \epsilon\left(t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial v}\right), \tag{4.6}$$

$$Y_2 = \left(\frac{\partial}{\partial t} + t\frac{\partial}{\partial v}\right) + \epsilon\left(t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial v}\right), \tag{4.7}$$

$$Y_3 = \left(\frac{\partial}{\partial x} + t\frac{\partial}{\partial v}\right) + \epsilon\left(x\frac{\partial}{\partial x} + (v+2x)\frac{\partial}{\partial v}\right). \tag{4.8}$$

The operators (4.6), (4.7) and (4.8) are linear combination of the generators X_2, X_3, X_7, X_8 and X_9 given in (3.39).

1. The operator

$$Y_1 = \epsilon\left(t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial v}\right)$$

has the following functionally independent invariants

$$\lambda_1 = x,$$

$$\lambda_2 = te^{\frac{v}{2x}}$$

and the corresponding approximate invariant solution is given by

$$v \approx 2x \ln\left(\frac{\varphi}{t}\right),$$

where φ satisfies the equation

$$\varphi'' + \frac{2}{x}\varphi' - \frac{(\varphi')^2}{\varphi} = \frac{e^{-2x\frac{\varphi'}{\varphi}}}{\varphi} + \frac{\epsilon A\varphi}{2\alpha x}.$$

2. The generator

$$Y_2 = \left(\frac{\partial}{\partial t} + t\frac{\partial}{\partial v}\right) + \epsilon\left(t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial v}\right)$$

has the following functionally independent invariants

$$\lambda_1 = x + \epsilon\phi_1\left(x, \frac{t^2}{2} - v\right),$$

$$\lambda_2 = \left(\frac{t^2}{2} - v\right) + \epsilon\phi_2\left(x, \frac{t^2}{2} - v\right).$$

Assuming that $\phi_1$and ϕ_2 are equal to zero, the corresponding approximate invariant solution is given by

$$v \approx \frac{t^2}{2} - \phi(x),$$

where ϕ satisfies the equation

$$\phi''(x) = \frac{1}{\alpha}\left(e^{\phi'(x)} - \epsilon A\right).$$

3. Using the generator

$$Y_3 = \left(\frac{\partial}{\partial x} + t\frac{\partial}{\partial v}\right) + \epsilon\left(x\frac{\partial}{\partial x} + (v + 2x)\frac{\partial}{\partial v}\right),$$

we obtain the following functionally independent invariants

$$\lambda_1 = t + \epsilon f\left(t, xt - v\right),$$

$$\lambda_2 = (xt - v) + \epsilon\left((xt - v + 2x) + g\left(t, xt - v\right)\right).$$

Assuming the functions f and g to be zero, the corresponding approximate invariant solution is given by

$$v \approx 2\epsilon - xt + 4\epsilon Ae^{-\frac{t}{2}} + t + c.$$

where A and c are constants.

Concluding Remarks

In this study a deeper understanding of the construction of the principal Lie algebra, the equivalence transformations, the approximate principal Lie algebra, the approximate equivalence transformations and the approximate invariant solutions has been gained. We have determined the function g for which the approximate principal Lie algebra extends by one and also we constructed some approximate invariant solutions for the equation (1.4).

Although not covered in this exercise, it would be interesting in the near future to extend this analysis to the equations

$$v_{tt} = \alpha v_x^{\sigma} v_{xx} + \epsilon g\left(v_x\right),$$

$$v_{tt} = \alpha v_x^{-4} v_{xx} + \epsilon g\left(v_x\right),$$

$$v_{tt} = \alpha v_x^{-\frac{4}{3}} v_{xx} + \epsilon g\left(v_x\right).$$

In section 4.1 the adjoint group for the L_{10} has been constructed, it could be interesting to find the optimal system of one-dimensional subalgebras of L_{10} and the invariant solutions.

The problem of finding the Lagrangians and conservation laws for the equations (1.4) is still to be solved. Moreover we wish to find some physical meaning or applications of these equations.

Appendix A Prolongation formulae

(a) We give explicit formulae for the prolongations (2.3) and (2.16). In the extended space with variables $(t, x, v, v_x, v_t, v_{xx}, v_{xt}, v_{tt})$ the prolonged operator is given by

$$\begin{aligned} X^{(2)} &= \xi^1(t,x,v)\frac{\partial}{\partial t} + \xi^2(t,x,v)\frac{\partial}{\partial x} + \eta(t,x,v)\frac{\partial}{\partial v} + \zeta_1\frac{\partial}{\partial v_t} + \zeta_2\frac{\partial}{\partial v_x} \\ &\quad + \zeta_{11}\frac{\partial}{\partial v_{tt}} + \zeta_{22}\frac{\partial}{\partial v_{xx}} + \zeta_{12}\frac{\partial}{\partial v_{xt}}, \end{aligned}$$

where

$$\zeta_1 = \eta_t + v_t\eta_v - v_t\xi^1_t - v_t^2\xi^1_v - v_x\xi^2_t - v_x v_t \xi^2_v$$

$$\zeta_2 = \eta_x + v_x\eta_v - v_t\xi^1_x - v_t v_x\xi^1_v - v_x\xi^2_x - v_x^2\xi^2_v,$$

$$\begin{aligned} \zeta_{11} &= \eta_{tt} + \left(2\eta_{tv} - \xi^1_{tt}\right)v_t - \xi^2_{tt}v_x + \left(\eta_v - 2\xi^1_t\right)v_{tt} - 2\xi^2_t v_{xt} + \left(\eta_{vv} - 2\xi^1_{tv}\right)v_t^2 \\ &\quad -2\xi^2_{tv}v_t v_x - \xi^1_{vv}v_t^3 - \xi^2_{vv}v_t^2 v_x - 3\xi^1_v v_t v_{tt} - \xi^2_v v_x v_{tt} - 2\xi^2_v v_t v_{xt}, \end{aligned}$$

$$\begin{aligned} \zeta_{22} &= \eta_{xx} + \left(2\eta_{xv} - \xi^2_{xx}\right)v_x - \xi^1_{xx}v_t + \left(\eta_v - 2\xi^2_x\right)v_{xx} - 2\xi^1_x v_{xt} + \left(\eta_{vv} - 2\xi^2_{xv}\right)v_x^2 \\ &\quad -2\xi^1_{xv}v_t v_x - \xi^2_{vv}v_x^3 - \xi^1_{vv}v_t v_x^2 - 3\xi^2_v v_x v_{xx} - \xi^1_v v_t v_{xx} - 2\xi^1_v v_x v_{xt}. \end{aligned}$$

(b) The extended generator of the equivalence transformations

$$\widehat{E} = E + \zeta_1\frac{\partial}{\partial v_t} + \zeta_2\frac{\partial}{\partial v_x} + \zeta_{11}\frac{\partial}{\partial v_{tt}} + \zeta_{22}\frac{\partial}{\partial v_{xx}} + \omega_1\frac{\partial}{\partial f_t} + \omega_2\frac{\partial}{\partial f_x} + \omega_3\frac{\partial}{\partial f_v} + \omega_4\frac{\partial}{\partial f_{v_t}},$$

where

$$\omega_1 = \widetilde{D}_t(\mu) - f_t\widetilde{D}_t\left(\xi^1\right) - f_x\widetilde{D}_t\left(\xi^2\right) - f_v\widetilde{D}_t(\eta) - f_{v_t}\widetilde{D}_t(\zeta_1) - f_{v_x}\widetilde{D}_t(\zeta_2),$$

$$\omega_2 = \widetilde{D}_x(\mu) - f_t\widetilde{D}_x\left(\xi^1\right) - f_x\widetilde{D}_x\left(\xi^2\right) - f_v\widetilde{D}_x(\eta) - f_{v_t}\widetilde{D}_x(\zeta_1) - f_{v_x}\widetilde{D}_x(\zeta_2),$$

$$\omega_3 = \widetilde{D}_v(\mu) - f_t\widetilde{D}_v\left(\xi^1\right) - f_x\widetilde{D}_v\left(\xi^2\right) - f_v\widetilde{D}_v(\eta) - f_{v_t}\widetilde{D}_v(\zeta_1) - f_{v_x}\widetilde{D}_v(\zeta_2),$$

$$\omega_4 = \widetilde{D}_{v_t}(\mu) - f_t\widetilde{D}_{v_t}\left(\xi^1\right) - f_x\widetilde{D}_{v_t}\left(\xi^2\right) - f_v\widetilde{D}_{v_t}(\eta) - f_{v_t}\widetilde{D}_{v_t}(\zeta_1) - f_{v_x}\widetilde{D}_{v_t}(\zeta_2).$$

The operators $\widetilde{D}_a$, $a \in \{t, x, v.v_t\}$ are given in p 9.

(c) The prolonged generator of approximate point transformations is given by

$$\begin{aligned}
\overline{E} \;=\; & \left(\xi^0 + \epsilon\xi^1\right)\frac{\partial}{\partial x} + \left(\tau^0 + \epsilon\tau^1\right)\frac{\partial}{\partial t} + \left(\eta^0 + \epsilon\eta^1\right)\frac{\partial}{\partial v} + \mu\frac{\partial}{\partial g} \\
& + \left(\zeta_0^x + \epsilon\zeta_1^x\right)\frac{\partial}{\partial v_x} + \left(\zeta_0^t + \epsilon\zeta_1^t\right)\frac{\partial}{\partial v_t} + \left(\zeta_0^{xx} + \epsilon\zeta_1^{xx}\right)\frac{\partial}{\partial v_{xx}} + \\
& \left(\zeta_0^{tt} + \epsilon\zeta_1^{tt}\right)\frac{\partial}{\partial v_{tt}} + \omega_0\frac{\partial}{\partial g_v} + \omega_1\frac{\partial}{\partial g_t} + \omega_2\frac{\partial}{\partial g_x} + \omega_{01}\frac{\partial}{\partial g_{v_t}} + \ldots,
\end{aligned}$$

where

$$\zeta_\nu^t = \eta_t^\nu + v_t\eta_v^\nu - v_t\tau_t^\nu - v_t^2\tau_v^\nu - v_x\xi_t^\nu - v_xv_t\xi_v^\nu, \qquad \nu = 0, 1,$$

$$\zeta_\nu^x = \eta_x^\nu + v_x\eta_v^\nu - v_t\tau_x^\nu - v_tv_x\tau_v^\nu - v_x\xi_x^\nu - v_x^2\xi_v^\nu, \qquad \nu = 0, 1,$$

$$\begin{aligned}
\zeta_\nu^{tt} \;=\; & \eta_{tt}^\nu + \left(2\eta_{tv}^\nu - \tau_{tt}^\nu\right)v_t - \xi_{tt}^\nu v_x + \left(\eta_v - 2\tau_t^\nu\right)v_{tt} - 2\xi_t^\nu v_{xt} + \left(\eta_{vv} - 2\tau_{tv}^\nu\right)v_t^2 \\
& -2\xi_{tv}^\nu v_tv_x - \tau_{vv}^\nu v_t^3 - \xi_{vv}^\nu v_t^2v_x - 3\tau_v^\nu v_tv_{tt} - \xi_v^\nu v_xv_{tt} - 2\xi_v^\nu v_tv_{xt}, \qquad \nu = 0, 1,
\end{aligned}$$

$$\begin{aligned}
\zeta_\nu^{xx} \;=\; & \eta_{xx}^\nu + \left(2\eta_{xv}^\nu - \xi_{xx}^\nu\right)v_x - \tau_{xx}^\nu v_t + \left(\eta_v^\nu - 2\xi_x^\nu\right)v_{xx} - 2\tau_x^\nu v_{xt} + \left(\eta_{vv}^\nu - 2\xi_{xv}^\nu\right)v_x^2 \\
& -2\tau_{xv}^\nu v_tv_x - \xi_{vv}^\nu v_x^3 - \tau_{vv}^\nu v_tv_x^2 - 3\xi_v^\nu v_xv_{xx} - \tau_v^\nu v_tv_{xx} - 2\tau_v^\nu v_xv_{xt}, \; \nu = 0, 1.
\end{aligned}$$

$$\begin{aligned}
\omega_0 \;=\; & \widetilde{D}_t(\mu) - g_t\widetilde{D}_t\left(\tau^0 + \epsilon\tau^1\right) - g_x\widetilde{D}_t\left(\xi^0 + \epsilon\xi^1\right) - g_v\widetilde{D}_t\left(\eta^0 + \epsilon\eta^1\right) - \\
& g_{v_t}\widetilde{D}_t\left(\zeta_0^t + \epsilon\zeta_1^t\right) - g_{v_x}\widetilde{D}_t\left(\zeta_0^x + \epsilon\zeta_1^x\right)
\end{aligned}$$

$$\begin{aligned}
\omega_1 \;=\; & \widetilde{D}_x(\mu) - g_t\widetilde{D}_x\left(\tau^0 + \epsilon\tau^1\right) - g_x\widetilde{D}_x\left(\xi^0 + \epsilon\xi^1\right) - g_v\widetilde{D}_x\left(\eta^0 + \epsilon\eta^1\right) - \\
& g_{v_t}\widetilde{D}_x\left(\zeta_0^t + \epsilon\zeta_1^t\right) - g_{v_x}\widetilde{D}_x\left(\zeta_0^x + \varepsilon\zeta_1^x\right)
\end{aligned}$$

$$\begin{aligned}\omega_2 &= \widetilde{D}_v(\mu) - g_t\widetilde{D}_v\left(\tau^0+\epsilon\tau^1\right) - g_x\widetilde{D}_v\left(\xi^0+\epsilon\xi^1\right) - g_v\widetilde{D}_v\left(\eta^0+\epsilon\eta^1\right) - \\ &\quad g_{v_t}\widetilde{D}_v\left(\zeta_0^t+\epsilon\zeta_1^t\right) - g_{v_x}\widetilde{D}_v\left(\zeta_0^x+\epsilon\zeta_1^x\right)\end{aligned}$$

$$\begin{aligned}\omega_{01} &= \widetilde{D}_{v_t}(\mu) - g_t\widetilde{D}_{v_t}\left(\tau^0+\epsilon\tau^1\right) - g_x\widetilde{D}_{v_t}\left(\xi^0+\epsilon\xi^1\right) - g_v\widetilde{D}_{v_t}\left(\eta^0+\epsilon\eta^1\right) - \\ &\quad g_{v_t}\widetilde{D}_{v_t}\left(\zeta_0^t+\epsilon\zeta_1^t\right) - g_{v_x}\widetilde{D}_{v_t}\left(\zeta_0^x+\epsilon\zeta_1^x\right)\end{aligned}$$

Appendix B

We show that the approximate symmetries obtained in (3.39) leave the equation (3.7) invariant. Consider the generators

$$X_9 = \epsilon\left(t\frac{\partial}{\partial t} - 2x\frac{\partial}{\partial v}\right) \quad \text{and} \quad X_{10} = \epsilon\left(x\frac{\partial}{\partial x} + (2x+v)\frac{\partial}{\partial v}\right).$$

We have

$$X_9^{(2)}\left(v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon g\left(v_x\right)\right)|_{v_{tt}=\alpha e^{v_x} v_{xx}} = \zeta_1^{tt} - \alpha e^{v_x}\zeta_1^{x} v_{xx} - \alpha e^{v_x}\zeta_1^{xx}|_{v_{tt}=\alpha e^{v_x} v_{xx}}, \quad (4.9)$$

where $X_9^{(2)}$ is the second prolongation of X_9, but

$$\zeta_1^t = v_t, \quad \zeta_1^{tt} = v_{tt}, \quad \zeta_1^x = 2, \quad \zeta_1^{xx} = -v_{xx},$$

hence the right hand side of equation (4.9) becomes

$$v_{tt} - 2\alpha e^{v_x} v_{xx} + \alpha v_{xx} e^{v_x}|_{v_{tt}=\alpha e^{v_x} v_{xx}} = 0.$$

Thus the equation (3.7) admits the generator X_9.

Similarly

$$X_{10}^{(2)}\left(v_{tt} = \alpha e^{v_x} v_{xx} + \epsilon g\left(v_x\right)\right)|_{v_{tt}=\alpha e^{v_x} v_{xx}} = \zeta_1^{tt} - \alpha e^{v_x}\zeta_1^{x} v_{xx} - \alpha e^{v_x}\zeta_1^{xx}|_{v_{tt}=\alpha e^{v_x} v_{xx}} \quad (4.10)$$

$$\zeta_1^t = -v_t, \quad \zeta_1^{tt} = -2v_{tt}, \quad \zeta_1^x = -2, \quad \zeta_1^{xx} = 0,$$

hence the right hand side of equation (4.10) becomes

$$-2v_{tt} + 2\alpha e^{v_x} v_{xx}|_{v_{tt}=\alpha e^{v_x} v_{xx}} = 0.$$

Thus the generator X_{10} leaves the equation (3.7) invariant.

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