ON THE SYMMETRY ANALYSIS OF SOME WAVE-TYPE NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

by

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Declaration

I declare that the thesis for the degree of Doctor of Philosophy at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

Signed: ....

MR GABRIEL MAGALAKWE

Date: ....

This thesis has been submitted with my approval as a University supervisor and would certify that the requirements for the applicable Doctor of Philosophy degree rules and regulations have been fulfilled.

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Declaration of Publications

Details of contribution to publications that form part of this thesis.

Chapter 2

G. Magalakwe, B. Muatjetjeja, C. M. Khalique, Symmetry analysis, nonlinearly self-adjoint and conservation laws of a generalized (2+1)-dimensional Klein-Gordon equation, Submitted for publication to Zeitschrift fuer Angewandte Mathematik und Physik.

Chapter 3

(i) G. Magalakwe, C. M. Khalique, New Exact Solutions for a Generalizes Double Sinh-Gordon Equation, Abstract and Applied Analysis, 2013, Article ID 268902 (2013).

(ii) G. Magalakwe, B. Muatjetjeja, C. M. Khalique, Generalized double sinh-Gordon equation: Symmetry reductions, exact solutions and conservation laws, Submitted for publication to Iranian Journal of Science and Technology.

Chapter 4

G. Magalakwe, B. Muatjetjeja, C. M. Khalique, Exact solutions and conservation laws for a generalized double combined sinh-cosh-Gordon equation, Submitted for publication to Hiroshima Mathematical Journal.

Chapter 5

G. Magalakwe, B. Muatjetjeja, C. M. Khalique, On the solutions and conservation laws for the (2+1)-dimensional nonlinear sinh-Gordon equation, Submitted for publication to Reports on Mathematical Physics.

Chapter 6

G. Magalakwe, B. Muatjetjeja, C. M. Khalique, Exact solutions and conservation laws for the (3+1)-dimensional nonlinear sinh-Gordon equation, To be Submitted for publication.

Chapter 7

G. Magalakwe, B. Muatjetjeja, C. M. Khalique, Exact solutions and conservation laws of the four Boussinesq-type equations, To be Submitted for publication.

Dedication

I dedicate this thesis to my late father Israel, whose memories motivate me as a person, and to my mother Alice for her motherly support and love. I also dedicate this work to my wife Nhlanhla for being there for me, and my child Gabriella. Lastly, I would like to dedicate this research work to my family and friends for keeping me going when it was tough.

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Abstract

In this work, we study the applications of Lie symmetry analysis to certain nonlinear wave equations. Exact solutions and conservation laws are obtained for such equations. The equations which are considered in this thesis are the generalized (2+1)-dimensional Klein-Gordon equation, the generalized double sinh-Gordon equation, the generalized double combined sinh-cosh-Gordon equation, the (2+1)-dimensional nonlinear sinh-Gordon equation, the (3+1)-dimensional nonlinear sinh-Gordon equation, the Boussinesq-double sine-Gordon equation, the Boussinesq-double sinh-Gordon equation, the Boussinesq-Liouville type I equation and the Boussinesq-Liouville type II equation.

The generalized (2+1)-dimensional Klein-Gordon equation is investigated from the point of view of Lie group classification. We show that this equation admits a nine-dimensional equivalence Lie algebra. It is also shown that the principal Lie algebra consists of six symmetries. Several possible extensions of the principal Lie algebra are computed and the group-invariant solutions of the generalized (2+1)-dimensional Klein-Gordon equation are presented for power law and exponential function cases. Thereafter, we illustrate that the generalized (2+1)-dimensional Klein-Gordon equation is nonlinearly self-adjoint. In addition, we derive conservation laws for the nonlinearly self-adjoint subclasses by using the new Ibragimov theorem.

Lie symmetry method is performed on a generalized double sinh-Gordon equation. Exact solutions of a generalized double sinh-Gordon equation are obtained by using the Lie symmetry method in conjunction with the simplest equation method and the exponential function method. In addition to exact solutions we also present conservation laws which are derived using four different methods, namely the direct method, the Noether theorem, the new conservation theorem due to Ibragimov and the multiplier method.

The generalized double combined sinh-cosh-Gordon equation is investigated using Lie group analysis. Exact solutions are obtained using the Lie group method together with the simplest equation method. Conservation laws are also obtained by using

four different approaches, namely the direct method, the Noether theorem, the new conservation theorem due to Ibragimov and the multiplier method for the underlying equation.

The (2+1)-dimensional nonlinear sinh-Gordon equation and the (3+1)-dimensional nonlinear sinh-Gordon equation are investigated by using Lie symmetry analysis. The similarity reductions and exact solutions with the aid of simplest equation method and (G'/G)-expansion methods are computed. In addition to exact solutions, the conservation laws are derived as well for both the equations.

Finally, the four Boussinesq-type equations, namely, the Boussinesq-double sine-Gordon equation, the Boussinesq-double sinh-Gordon equation, the Boussinesq-Liouville type I equation and the Boussinesq-Liouville type II equation are analysed using Lie group analysis. Exact solutions for these equations are obtained using the Lie symmetry method in conjunction with the simplest equation. Conservation laws are also obtained for these equations by employing two methods, namely, the Noether theorem and the multiplier method.

List of Acronyms

DEs:	Differential equations
ODEs:	Ordinary differential equations
PDEs:	Partial differential equations
NLPDEs:	Nonlinear partial differential equations

Introduction

It is well known that finding exact travelling wave solutions of nonlinear partial differential equations (NLPDEs) is important in many scientific areas such as fluid mechanics, plasma physics and quantum field theory. Due to these applications many researchers are investigating exact solutions of NLPDEs, since they play a vital role in the study of nonlinear physical phenomena. Finding exact solutions of such NLPDEs provides us with a better understanding of the physical phenomena that these NLPDEs describe. Several techniques have been presented in the literature to find exact solutions of the NLPDEs. These include: the inverse scattering transform method [1], the variable separated ODE method [2], the Darboux transformation method [3], the homogeneous balance method [4], the Weierstrass elliptic function expansion method [5], the F-expansion method [6], the (G'/G)-expansion method [7, 8], the exponential function method [9, 10], the tanh function method [11–13], the extended tanh function method [14], the sine-cosine method [15], the bifurcation method [16] and the Lie symmetry method [17].

The Lie symmetry method is based on symmetry and invariance principles and is a systematic method for solving differential equations analytically. There is no doubt that Lie symmetry method is one of the most powerful methods to determine solutions of NLPDEs. It was first developed by Sophus Lie (1842-1899) in the late nineteenth century. In recent years, this method has become an essential tool for anyone investigating mathematical models of physical, engineering and natural problems. Several good books are available on this subject. See for example, [17–24].

Many differential equations of physical interest involve parameters, arbitrary elements or functions, which need to be determined. The construction of different forms of these parameters is one of the most essential tasks in nonlinear science. Usually the various forms of these parameters are determined from experiments. However, the Lie symmetry approach through the method of group classification [22–31] has proven to be a versatile tool in specifying different forms of these parameters systematically. The first group classification problem was investigated by Sophus Lie [25] in 1881 for a linear second-order partial differential equations with two independent variables. The main concept of group classification of a differential equation involving arbitrary element, say, for example, $p(u)$, consists of finding the Lie point symmetries of the differential equation with arbitrary function $p(u)$, and then computing systematically all possible forms of $p(u)$ which extend the principal Lie algebra.

The notion of conservation laws plays an important role in the solution process and reduction of differential equations. Conservation laws are mathematical expressions of the physical laws, such as conservation of energy, mass, momentum and so on. In the literature, conservation laws have been extensively used in various aspects (see for example [32–40]). For example exact solutions of some partial differential equations have been obtained using conserved vectors associated with the Lie point symmetries [37, 38, 40]. The celebrated Noether theorem [41] provides an elegant and constructive way of obtaining conserved vectors. In fact, it provides an explicit formula for determining a conservation law once a Noether symmetry corresponding to the Lagrangian is known for an Euler-Lagrange equation. Also conservation laws were used in the numerical integration of partial differential equations [42], for instance, to control numerical errors. Comparison of different approaches to construct conservation laws of partial differential equations can be found in [43].

In this thesis we explore the application of symmetry analysis [22–24] by studying nine NLPDEs. The nine NLPDEs that will be studied are the generalized (2+1)-dimensional Klein-Gordon equation, the generalized double sinh-Gordon equation, the generalized double combined sinh-cosh-Gordon equation, the (2+1)-dimensional nonlinear sinh-Gordon equation, the (3+1)-dimensional nonlinear sinh-Gordon equation, the Boussinesq-double sine-Gordon equation, the Boussinesq-double sinh-Gordon equation, the Boussinesq-Liouville type I equation and the Boussinesq-Liouville type II equation.

Firstly, this thesis considers a generalized (2+1)-dimensional Klein-Gordon equation, given by

$$u_{tt} - u_{xx} - u_{yy} + p(u) = 0, \tag{1}$$

where $p(u)$ is an arbitrary function u. Equation (1) is one of the equations which describes nonlinear wave motion and has many scientific applications in solid state physics, nonlinear optics, plasma physics and fluid dynamics.

Secondly, we study the generalized double sinh-Gordon equation [13, 16, 44] given by

$$u_{tt} - ku_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu) = 0, \tag{2}$$

where k, α and β are non-zero real constants and n is a positive integer. Here u is a real scalar function of the two independent variables x and t. This equation arises in a wide range of scientific applications that range from chemical reactions to water surface gravity waves.

Thirdly, the equation that is studied in this thesis is a generalized form of the double combined sinh-cosh-Gordon equation [2, 7, 45] given by

$$u_{tt} - ku_{xx} + \alpha \sinh(nu) + \alpha \cosh(nu) + \beta \sinh(2nu) + \beta \cosh(2nu) = 0, \qquad n \geq 1, \tag{3}$$

where k, α and β are non-zero constants. Here $u(t, x)$ is a function of space x and time variable t. Equation (3) is well known NLPDE which admits geometric interpretation as the differential equation which determines time-like surfaces of constant positive curvature in the same spaces and it also combines the effect of sine and cosine hyperbolic terms.

The (1+1)-dimensional sinh-Gordon equation [46]

$$u_{tt} - u_{xx} + \sinh u = 0, \tag{4}$$

which is widely used in mathematical physics and engineering sciences is a nonlinear hyperbolic partial differential equation. Equation (4) usually describes water waves, the vibration of a string or a membrane, the propagation of electromagnetic and sound waves or the transmission of electric signals in a cable. The sinh-Gordon equation first appeared in the propagation of fluxons in Josephson junctions [47] between two superconductors and it started to attract lot of attention in the 1970s due to the presence of soliton solutions. The sinh-Gordon equation appears also in (2+1) dimensions and (3+1) dimensions.

The (2+1)-dimensional sinh-Gordon [46]

$$u_{tt} - u_{xx} - u_{yy} + \sinh u = 0, \tag{5}$$

which plays an important role in nonlinear science such as solid state physics, fluid dynamics, integrable field theory and nonlinear optics will be the fourth equation that will be studied in this thesis. Here $u(t, x, y)$ is a function of space x, y and time variable t.

Next we study the (3+1)-dimensional sinh-Gordon [46]

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} + \sinh u = 0, \tag{6}$$

where $u(t, x, y, z)$ is a function of space variables x, y, z and time variable t. This equation also appears in solid state physics, fluid dynamics, integrable field theory, nonlinear optics and it has applications in many areas of physics.

Lastly, the Boussinesq-double sine-Gordon equation, the Boussinesq-double sinh-Gordon equation, the Boussinesq-Liouville type I equation and the Boussinesq-Liouville type II equation [48], given by

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sin u + \frac{3}{2}\sin 2u, \tag{7}$$

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sinh u + \frac{3}{2}\sinh 2u, \tag{8}$$

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = e^{u} + \frac{3}{4}\sinh e^{2u} \tag{9}$$

and

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = e^{-u} + \frac{3}{4}e^{-2u}, \tag{10}$$

which appear in a diverse range of areas of physics will be investigated. These equations have applications in scientific fields such as solid state physics, non-linear optics and fluid motion.

The outline of this thesis is as follows.

In Chapter one, the basic definitions and theorems concerning the one-parameter groups of transformations and conservation laws are presented.

Chapter two deals with the generalized (2+1)-dimensional Klein-Gordon equation (1).

Chapters three and four discuss the solutions and conservation laws of a generalized double sinh-Gordon equation (2) and the generalized double combined sinh-cosh-Gordon equation (3), respectively.

Chapters five and six deal with the solutions and conservation laws of a (2+1)-dimensional nonlinear sinh-Gordon equation (5) and the (3+1)-dimensional nonlinear sinh-Gordon equation (6), respectively.

Chapter seven discusses the solutions and conservation laws of the Boussinesq-double sine-Gordon equation (7), the Boussinesq-double sinh-Gordon equation (8), the Boussinesq-Liouville type I equation (9) and the Boussinesq-Liouville type II equation (10), respectively.

Finally, in Chapter eight, a summary of the results of the thesis is presented and future work is discussed.

Bibliography is given at the end.

Chapter 1

Preliminaries

In this chapter we give some basic methods of Lie symmetry analysis and conservation laws of partial differential equations (PDEs).

1.1 Introduction

In the late nineteenth century an outstanding mathematician Sophus Lie (1842-1899) developed a new method, known as Lie group analysis, for solving differential equations and showed that the majority of adhoc methods of integration of differential equations could be explained and deduced simply by means of his theory. Recently, many good books have appeared in the literature in this field. We mention a few here, Ovsiannikov [17], Stephani [18], Ibragimov [19,20], Cantwell [21], Bluman and Kumei [22], and Olver [23]. Definitions and results given in this Chapter are taken from the books mentioned above.

Conservation laws for PDEs are constructed here using four different approaches; the direct method [39], the Noether theorem [41], the new conservation theorem due to Ibragimov [49], and the multiplier method [50]. First we present some preliminaries which we will need later in the thesis. For details the reader is referred to [24,39,41, 49,50].

1.2 Continuous one-parameter groups

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $u = (u^1, ..., u^m)$ be the dependent variables with coordinates u^α (n and m finite). Consider a change of the variables x and u involving a real parameter a:

$$T_a : \bar{x}^i = f^i(x, u, a), \ \ \bar{u}^\alpha = \phi^\alpha(x, u, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and u if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \phi(a, b)), \\ \bar{\bar{u}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \ \text{ where } w(s) = \left.\frac{\partial\, \phi(s, b)}{\partial b}\right|_{b=0}.$$

1.3 Prolongation of point transformations and group generator

The derivatives of u with respect to x are defined as

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives u_i^α is denoted by $u_{(1)}$, i.e.,

$$u_{(1)} = \{u_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$u_{(2)} = \{u_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $u_{(3)} = \{u_{ijk}^\alpha\}$ and likewise $u_{(4)}$ etc. Since $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$ contains only u_{ij}^α for $i \leq j$. In the same manner $u_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $u_{(4)}, u_{(5)} \cdots$.

In group analysis all variables $x, u, u_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the u_s^α are called differential variables [24].

We now consider a pth-order PDE(s), namely

$$E_\alpha(x, u, u_{(1)}, ..., u_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, u)$, one-parameter group of transformations G is

$$\bar{x}^i = f^i(x, u, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{u^\alpha} = \phi^\alpha(x, u, a), \quad \phi^\alpha|_{a=0} = u^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x^i} \approx x^i + a\,\xi^i(x, u), \quad \bar{u^\alpha} \approx u^\alpha + a\,\eta^\alpha(x, u) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = u^\alpha.$$

Thus, we have

$$\xi^i(x, u) = \frac{\partial f^i}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x, u) = \frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{u^\alpha} \approx (1 + a\,X)u,$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{u}_i^\alpha = \bar{D}_j(u^\alpha), \qquad \bar{u}_{ij}^\alpha = \bar{D}_j(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots.$$

Now let us apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha) \\ &= D_i(f^j)\bar{u}^\alpha_j. \end{aligned} \tag{1.11}$$

This

$$\left(\frac{\partial f^j}{\partial x^i} + u^\beta_i \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}^\alpha_j = \frac{\partial \phi^\alpha}{\partial x^i} + u^\beta_i \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{1.12}$$

The quantities $\bar{u}^\alpha_j$ can be represented as functions of $x, u, u_{(i)}, a$ for small a, ie., (1.12) is locally invertible:

$$\bar{u}^\alpha_i = \psi^\alpha_i(x, u, u_{(1)}, a), \quad \psi^\alpha|_{a=0} = u^\alpha_i. \tag{1.13}$$

The transformations in $x, u, u_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{u}^\alpha_i \approx u^\alpha_i + a\zeta^\alpha_i \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (1.7) and (1.14).

Higher-order prolongations of G, viz., $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (1.7) and (1.14) we get

$$\begin{aligned} D_i(f^j)(\bar{u}^\alpha_j) &= D_i(\phi^\alpha)\,. \\ D_i(x^j + a\xi^j)(u^\alpha_j + a\zeta^\alpha_j) &= D_i(u^\alpha + a\eta^\alpha) \\ (\delta^j_i + aD_i\xi^j)(u^\alpha_j + a\zeta^\alpha_j) &= u^\alpha_i + aD_i\eta^\alpha \\ u^\alpha_i + a\zeta^\alpha_i + au^\alpha_j D_i\xi^j &= u^\alpha_i + aD_i\eta^\alpha \end{aligned}$$

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad \text{(sum on } j). \tag{1.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - u_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k). \tag{1.16}$$

By induction (recursively)

$$\zeta_{i_1,i_2,\dots,i_p}^\alpha = D_{i_p}(\zeta_{i_1,i_2,\dots,i_{p-1}}^\alpha) - u_{i_1,i_2,\dots,i_{p-1}j}^\alpha D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$X^{[1]} = X + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} \quad \text{(sum on } i, \alpha),$$

$$\vdots$$

$$X^{[p]} = X^{[p-1]} + \zeta_{i_1,\dots,i_p}^\alpha \frac{\partial}{\partial u_{i_1,\dots,i_p}^\alpha} \quad p \geq 1,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

1.4 Group admitted by a partial differential equation

Definition 1.2 The vector field

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{1.18}$$

is a *point symmetry* of the pth-order PDE (1.5), if

$$X^{[p]}(E_\alpha) = 0 \tag{1.19}$$

whenever $E_\alpha = 0$. This can also be written as

$$X^{[p]} E_\alpha\big|_{E_\alpha=0} = 0, \tag{1.20}$$

where the symbol $|_{E_\alpha=0}$ means evaluated on the equation $E_\alpha = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\alpha(\bar{x}, \bar{u}, \bar{u}_{(1)}, \cdots, \bar{u}_{(p)}) = 0, \tag{1.21}$$

where the function E_α is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, u)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{u}) \equiv F(f^i(x, u, a),\ \phi^\alpha(x, u, a)) = F(x, u), \tag{1.22}$$

identically in x, u and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, u)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial F}{\partial u^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has $n-1$ functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, u) = c_1, \cdots, J_{n-1}(x, u) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^n}{\eta^n(x,u)}.$$

Theorem 1.3 If the infinitesimal transformation (1.7) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{u}^\alpha\big|_{a=0} = u\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u}$ with the following property. If the operators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u}, \quad X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Conservation laws

1.7.1 Fundamental operators and their relationship

Let us consider a pth-order system of PDEs

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(p)}) = 0, \quad \alpha = 1, \dots, m, \tag{1.25}$$

of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$. Here $u_{(1)}, u_{(2)}, \dots, u_{(p)}$ denote the collections of all first, second, ..., pth-order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$, respectively, with the *total derivative operator* with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots, \quad i = 1, \dots, n, \tag{1.26}$$

and the summation convention is used whenever appropriate.

The *Euler-Lagrange operator*, for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^\alpha}, \quad \alpha = 1, \dots, m, \tag{1.27}$$

and the *Lie-Bäcklund operator* is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}, \ \xi^i, \eta^\alpha \in \mathcal{A}, \tag{1.28}$$

where $\mathcal{A}$ is the space of *differential functions*. The operator (1.28) is an abbreviated form of infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta^\alpha_{i_1 i_2 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}, \tag{1.29}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij}, \\ \zeta^\alpha_{i_1 \ldots i_s} &= D_{i_1} \ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{j i_1 \ldots i_s}, \quad s > 1, \end{aligned} \tag{1.30}$$

in which W^α is the *Lie characteristic function* defined by

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j. \tag{1.31}$$

The Lie-Bäcklund operator (1.29) in characteristic form can be written as

$$X = \xi^i D_i + W^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\partial}{\partial u^\alpha_{i_1 i_2 \ldots i_s}}. \tag{1.32}$$

The *Noether operators* corresponding to the Lie-Bäcklund symmetry operator X are given by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u^\alpha_i} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{i i_1 i_2 \ldots i_s}}, \quad i = 1, \ldots, n, \tag{1.33}$$

where the Euler-Lagrange operators with respect to derivatives of u^α are obtained from (1.27) by substituting u^α by the corresponding derivatives. For example,

$$\frac{\delta}{\delta u^\alpha_i} = \frac{\partial}{\partial u^\alpha_i} + \sum_{s \geq 1} (-1)^s D_{j_1} \ldots D_{j_s} \frac{\partial}{\partial u^\alpha_{i j_1 j_2 \ldots j_s}}, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m, \tag{1.34}$$

and the Euler-Lagrange , Lie-Bäcklund and Noether operators are connected by the operator identity

$$X + D_i(\xi^i) = W^\alpha \frac{\delta}{\delta u^\alpha} + D_i N^i. \tag{1.35}$$

The n-tuple vector $T = (T^1, T^2, \ldots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \ldots, n$, is a *conserved vector* of (1.25) if T^i satisfies

$$D_i T^i|_{(1.25)} = 0. \tag{1.36}$$

The equation (1.36) defines a *local conservation law* of system (1.25).

A Lie-Bäcklund operator X is said to be a *Noether symmetry* generator associated with a Lagrangian $L \in \mathcal{A}$ if there exists a vector $B = (B^1, ..., B^n), B^i \in \mathcal{A}$, such that

$$XL + LD_i(\xi^i) = D_i(B^i). \tag{1.37}$$

We now recall the Noether theorem [41].

Noether theorem [41]. For any Noether symmetry generator X associated with a given Lagrangian $L \in \mathcal{A}$, there corresponds a vector $T = (T^1, \ldots, T^n)$, $T^i \in \mathcal{A}$, given by

$$T^i = N^i(L) - B^i, \quad i = 1, ..., n, \tag{1.38}$$

which is a conserved vector of the Euler-Lagrange equations $\delta L/\delta u^\alpha = 0$, $\alpha = 1, ..., m$, where $\delta/\delta u^\alpha$ is the Euler-Lagrange operator given by (1.27) and the Noether operator associated with X is defined by (1.33) in which the Euler-Lagrange operators with respect to derivatives of u^α are obtain from (1.27) by substituting u^α by the corresponding derivatives.

1.7.2 Variational method for a system and its adjoint

The system of *adjoint equations* to the system of pth-order differential equations (1.25) is given by [51]

$$E^*_\alpha(x, u, v, \ldots, u_{(p)}, v_{(p)}) = 0, \quad \alpha = 1, \ldots, m, \tag{1.39}$$

where

$$E^*_\alpha(x, u, v, \ldots, u_{(p)}, v_{(p)}) = \frac{\delta(v^\beta E_\beta)}{\delta u^\alpha}, \quad \alpha = 1, \ldots, m, \; v = v(x) \tag{1.40}$$

and $v = (v^1, v^2, \ldots, v^m)$ are new dependent variables.

We also recalled the following results as presented in Ibragimov [49].

A system of equations (1.25) is said to be *self-adjoint* if the substitution of $v = u$ into the system of adjoint equations (1.39) yields the same system (1.25).

Equation (1.25) is said to be nonlinearly self-adjoint if the equation obtained from the adjoint equation (1.39) by the substitution $v = h(x, t, u, u_{(1)}, \dots)$, with a certain function $h(x, t, u, u_{(1)}, \dots)$ such that $h(x, t, u, u_{(1)}, \dots) \neq 0$, is identical to the original equation (1.25).

Suppose the system of equations (1.25) admits the symmetry operator

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha}. \tag{1.41}$$

Then the system of adjoint equations (1.39) admits the operator

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta^\alpha_* \frac{\partial}{\partial v^\alpha}, \quad \eta^\alpha_* = -[\lambda^\alpha_\beta v^\beta + v^\alpha D_i(\xi^i)], \tag{1.42}$$

where the operator (1.42) is an extension of (1.41) to the variable v^α and the λ^α_β are obtainable from

$$X(E_\alpha) = \lambda^\beta_\alpha E_\beta. \tag{1.43}$$

The following theorem is taken from [49].

Theorem 1.7.1 *Every Lie point, Lie-Bäcklund and non local symmetry* (1.41) *admitted by the system of equations* (1.25) *gives rise to a conservation law for the system consisting of the equation* (1.25) *and the adjoint equation* (1.39), *where the components T^i of the conserved vector $T = (T^1, \dots, T^n)$ are determined by*

$$\begin{aligned} T^i = \; & \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) - \cdots \right] \\ & + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_{ij}} - D_k \left(\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} \right) + \cdots \right] + D_j D_k(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u^\alpha_{ijk}} - \cdots \right] + \cdots , \end{aligned} \tag{1.44}$$

with Lagrangian given by

$$\mathcal{L} = v^\alpha \, E_\alpha \left(x, u, u_{(1)}, \dots, u_{(p)} \right). \tag{1.45}$$

The *multiplier method* used to construct conservation laws can be found in [23,50,52]. A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \ldots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{1.46}$$

hold identically. Here we will consider the first order multipliers, viz., $\Lambda_\alpha = \Lambda_\alpha(t, x, u, u_t, u_x)$. The right hand side of (1.46) is a divergence expression. The determining equation for the multiplier Λ_α is

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{1.47}$$

Once the multipliers are obtained the conserved vectors are constructed by invoking the homotopy formula [50].

1.8 Conclusion

In this chapter we presented a brief introduction to the Lie group analysis and conservation laws of PDEs and gave some results which will be used throughout this thesis. We also gave the algorithm to determine the Lie point symmetries and conservation laws of PDEs.

Chapter 2

Symmetry analysis, nonlinearly self-adjoint and conservation laws of a generalized (2+1)-dimensional Klein-Gordon equation

2.1 Introduction

This chapter aims to study a generalized Klein-Gordon equation in (2+1) dimensions, given by

$$u_{tt} - u_{xx} - u_{yy} + p(u) = 0, \tag{2.1}$$

where $p(u)$ is an arbitrary function of u. Firstly, we carry out Lie group classification of equation (2.1). We then find exact solutions of certain cases of the arbitrary element $p(u)$. Lastly, we construct conservation laws for the nonlinearly self-adjoint subclass of the generalized (2+1)-dimensional Klein-Gordon equation.

This work is new and has been submitted for publication. See [53].

2.2 Equivalence transformations

An equivalence transformation (see for example [24]) of (2.1) is an invertible transformation involving the independent variables t, x, y and dependent variable u that maps (2.1) into itself. The operator

$$\begin{aligned} Y &= \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \psi(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u \\ &\quad +\mu(t,x,y,u,p)\partial_p \end{aligned} \tag{2.2}$$

is the generator of the equivalence group for equation (2.1) provided it is admitted by the extended system

$$u_{tt} - u_{xx} - u_{yy} + p(u) = 0, \tag{2.3a}$$

$$p_t = 0, \quad p_x = 0, \quad p_y = 0. \tag{2.3b}$$

The prolonged operator for the extended system (2.3) has the form

$$\widetilde{Y} = Y^{[2]} + \omega_t \partial_{p_t} + \omega_x \partial_{p_x} + \omega_y \partial_{p_y} + \omega_u \partial_{p_u}, \tag{2.4}$$

where $Y^{[2]}$ is the second-prolongation of (2.2) given by

$$\begin{aligned} Y^{[2]} &= \tau\partial_t + \xi\partial_x + \psi\partial_y + \eta\partial_u + \mu\partial_p + \zeta_{tt}\partial_{u_{tt}} + \zeta_{xx}\partial_{u_{xx}} \\ &\quad +\zeta_{yy}\partial_{u_{yy}}. \end{aligned}$$

The coefficients ζ's and ω's are defined by the prolongation formulae

$$\begin{aligned} \zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi) - u_y D_t(\psi), \\ \zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\psi), \\ \zeta_y &= D_y(\eta) - u_t D_y(\tau) - u_x D_y(\xi) - u_y D_y(\psi), \\ \zeta_{tt} &= D_t(\zeta_x) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi) - u_{ty} D_t(\psi), \\ \zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\psi), \\ \zeta_{yy} &= D_y(\zeta_y) - u_{ty} D_y(\tau) - u_{xy} D_y(\xi) - u_{yy} D_y(\psi), \end{aligned}$$

and

$$\omega_t = \widetilde{D}_t(\mu) - p_t \widetilde{D}_t(\tau) - p_x \widetilde{D}_t(\xi) - p_y \widetilde{D}_t(\psi) - p_u \widetilde{D}_t(\eta),$$

$$\omega_x = \widetilde{D}_x(\mu) - p_t\widetilde{D}_x(\tau) - p_x\widetilde{D}_x(\xi) - p_y\widetilde{D}_x(\psi) - p_u\widetilde{D}_x(\eta),$$
$$\omega_y = \widetilde{D}_y(\mu) - p_t\widetilde{D}_y(\tau) - p_x\widetilde{D}_y(\xi) - p_y\widetilde{D}_y(\psi) - p_u\widetilde{D}_y(\eta),$$
$$\omega_u = \widetilde{D}_u(\mu) - p_t\widetilde{D}_u(\tau) - p_x\widetilde{D}_u(\xi) - p_y\widetilde{D}_u(\psi) - p_u\widetilde{D}_u(\eta),$$

respectively, where

$$D_t = \partial_t + u_t\partial_u + \cdots, \quad D_x = \partial_x + u_x\partial_u + \cdots, \quad D_y = \partial_y + u_y\partial_u + \cdots$$

are the total derivative operators and

$$\widetilde{D}_t = \partial_t + p_t\partial_p + \cdots,$$
$$\widetilde{D}_x = \partial_x + p_x\partial_p + \cdots,$$
$$\widetilde{D}_y = \partial_y + p_y\partial_p + \cdots,$$
$$\widetilde{D}_u = \partial_u + p_u\partial_p + \cdots$$

are the total derivative operators for the extended system. The application of the prolongation (2.4) and the invariance conditions of system (2.3) leads to the following equivalent generators:

$$Y_1 = \partial_t,\ Y_2 = \partial_x,\ Y_3 = \partial_y,\ Y_4 = \partial_u,\ Y_5 = -y\partial_x + x\partial_y,\ Y_6 = x\partial_t + t\partial_x,$$
$$Y_7 = y\partial_t + t\partial_y,\ Y_8 = u\partial_u + p\partial_p,\ Y_9 = x\partial_x + y\partial_y + t\partial_t - 2p\partial_p.$$

Thus the nine-parameter equivalence group is given by

$$Y_1 \ :\ \bar{t} = t + a_1,\ \bar{x} = x,\ \bar{y} = y,\ \bar{u} = u,\ \bar{p} = p,$$
$$Y_2 \ :\ \bar{t} = t,\ \bar{x} = x + a_2,\ \bar{y} = y,\ \bar{u} = u,\ \bar{p} = p,$$
$$Y_3 \ :\ \bar{t} = t,\ \bar{x} = x,\ \bar{y} = y + a_3,\ \bar{u} = u,\ \bar{p} = p,$$
$$Y_4 \ :\ \bar{t} = t,\ \bar{x} = x,\ \bar{y} = y,\ \bar{u} = u + a_4,\ \bar{p} = p,$$
$$Y_5 \ :\ \bar{t} = t,\ \bar{x} = x - a_5 y,\ \bar{y} = y + a_5 x,\ \bar{u} = u,\ \bar{p} = p,$$
$$Y_6 \ :\ \bar{t} = t + a_6 x,\ \bar{x} = x + a_6 t,\ \bar{y} = y,\ \bar{u} = u +\ \bar{p} = p,$$
$$Y_7 \ :\ \bar{t} = t + a_7 y,\ \bar{x} = x,\ \bar{y} = y + a_7 t,\ \bar{u} = u,\ \bar{p} = p,$$
$$Y_8 \ :\ \bar{t} = t,\ \bar{x} = x,\ \bar{y} = y,\ \bar{u} = ue^{a_8},\ \bar{p} = pe^{a_8},$$
$$Y_9 \ :\ \bar{t} = te^{a_9},\ \bar{x} = xe^{a_9},\ \bar{y} = ye^{a_9},\ \bar{u} = u,\ \bar{p} = pe^{-2a_9}.$$

and their composition gives

$$\begin{aligned}
\bar{t} &= (t + a_1 + a_6x + a_7y)e^{a_9}, \\
\bar{x} &= (x + a_2 - a_5y + a_6t)e^{a_9}, \\
\bar{y} &= (y + a_3 + a_5x + a_7t)e^{a_9}, \\
\bar{u} &= (u + a_4)e^{a_8}, \\
\bar{p} &= pe^{a_8 - 2a_9}.
\end{aligned}$$

2.3 Principal Lie algebra

The symmetry group of equation (2.1) will be generated by the vector field of the form

$$\Gamma = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \psi(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u. \quad (2.5)$$

The application of the second prolongation of Γ to (2.1) yields the following overdetermined system of linear partial differential equations (PDEs):

$$\begin{aligned}
&\tau_u = 0,\ \xi_u = 0,\ \psi_u = 0,\ \eta_{uu} = 0,\ \tau_y - \psi_t = 0,\ \xi_y + \psi_x = 0,\ \xi_t - \tau_x = 0, \\
&\psi_y - \tau_t = 0,\ \psi_y - \xi_x = 0,\ \tau_{tt} - \tau_{xx} - \tau_{yy} + 2\eta_{tu} = 0, \\
&\xi_{tt} - \xi_{xx} - \xi_{yy} + 2\eta_{xu} = 0,\ \psi_{tt} - \psi_{xx} - \psi_{yy} + 2\eta_{yu} = 0, \qquad (2.6) \\
&p(u)\eta_u - 2p(u)\psi_y - p'(u)\eta - \eta_{tt} + \eta_{xx} + \eta_{xx} = 0.
\end{aligned}$$

Solving the above system for arbitrary p, we find that the principal Lie algebra consists of six operators, namely

$$\begin{aligned}
&\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \partial_y, \quad \Gamma_4 = y\partial_t + t\partial_y, \\
&\Gamma_5 = x\partial_t + t\partial_x, \quad \Gamma_6 = y\partial_x - x\partial_y.
\end{aligned}$$

2.4 Lie group classification

Solving system (2.6), we obtain the classifying relation

$$(u\beta + \gamma)p'(u) + \alpha p(u) + \lambda = 0,$$

where β, γ, α and λ are constants. This classifying relation is invariant under the equivalence transformations of Section 2.2 if

$$\bar{\beta} = \beta, \; \bar{\gamma} = a_4\beta + \gamma e^{-a_8}, \; \bar{\alpha} = \alpha, \; \bar{\lambda} = \lambda e^{2a_9 - a_8}. \tag{2.7}$$

The above relation leads to the following five cases for the function p. For each case, we also provide the associated extended symmetries.

Case 1 $p(u)$ arbitrary but not of the form in Cases $2-5$.

In this case, we obtain the principal Lie algrebra, viz.,

$$\begin{aligned} &\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \partial_y, \quad \Gamma_4 = y\partial_t + t\partial_y, \\ &\Gamma_5 = x\partial_t + t\partial_x, \quad \Gamma_6 = y\partial_x - x\partial_y. \end{aligned} \tag{2.8}$$

Case 2 $p(u) = \sigma + \delta u$, where σ and δ are constants.

Here two subcases arise:

2.1 $\sigma, \delta \neq 0$.

The corresponding equation (2.1) extends the principal Lie algebra by

$$\Gamma_7 = u\partial_u, \; \Gamma_8 = F\,(t,x,y)\,\partial_u,$$

where $F(t,x,y)$ is any solution of

$$F_{tt} - F_{xx} - F_{yy} + \delta F - C_1\sigma = 0$$

and C_1 is a constant.

2.2 $\sigma \neq 0$, $\delta = 0$.

This subcase extends the principal Lie algebra by six symmetries

$$\begin{aligned} &\Gamma_7 = u\partial_u, \; \Gamma_8 = (t^2 + x^2 + y^2)\partial_t + 2tx\partial_x + ty\partial_y - tu\partial_u, \\ &\Gamma_9 = t\partial_t + x\partial_x + y\partial_y, \; \Gamma_{10} = 2ty\partial_t + 2yx\partial_x + (t^2 - x^2 + y^2)\partial_y - yu\partial_u, \\ &\Gamma_{11} = 2tx\partial_t + (t^2 + x^2 - y^2)\partial_x + 2xy\partial_y - xu\partial_u, \; \Gamma_{12} = F\,(t,x,y)\,\partial_u, \end{aligned}$$

where $F(t,x,y)$ is any solution of

$$2F_{tt} - 2F_{xx} - 2F_{yy} + 10C_4\sigma t + 5C_6\sigma x - 10C_7\sigma y - 2C_1\sigma + 5C_{11}\sigma = 0$$

and $C_1, C_4, C_6, C_7, C_{11}$ are arbitrary constants.

Case 3 $p(u) = \sigma + \delta u^n$, where σ is a constant, δ is non-zero constant and $n \neq 0, 1$.

Three subcases arise. These are

3.1 $\sigma \neq 0$.

In this subcase we have no additional Lie point symmetry.

3.2 $\sigma = 0$, $n \neq 5$.

Here the principal Lie algebra is extended by one symmetry

$$\Gamma_7 = (n-1)t\partial_t + (n-1)x\partial_x + (n-1)y\partial_y - 2u\partial_u.$$

3.3 $\sigma = 0$, $n = 5$.

In this subcase, the Lie point symmetries that extend the principal Lie algebra are

$$\begin{aligned}
\Gamma_7 &= (t^2 + x^2 + y^2)\partial_t + 2tx\partial_x + ty\partial_y - tu\partial_u, \\
\Gamma_8 &= 2ty\partial_t + 2yx\partial_x + (t^2 - x^2 + y^2)\partial_y - yu\partial_u, \\
\Gamma_9 &= 2tx\partial_t + (t^2 + x^2 - y^2)\partial_x + 2xy\partial_y - xu\partial_u, \\
\Gamma_{10} &= 2t\partial_t + 2x\partial_x + 2y\partial_y - u\partial_u.
\end{aligned}$$

Case 4 $p(u) = \sigma + \delta e^{nu}$, where σ is a constant, δ and n are non-zero constants.

Here two subcases arise.

4.1 $\sigma \neq 0$.

There is no extension of the principal Lie algebra in this subcase.

4.2 $\sigma = 0$.

The extra Lie point symmetry is

$$\Gamma_7 = nt\partial_t + nx\partial_x + ny\partial_y - 2\partial_u.$$

Remark. In subcases 4.2 we retrieve two special equations, namely, the Liouville equation in (2+1) dimensions [54] $u_{tt} - u_{xx} - u_{yy} + \delta e^{nu} = 0$ and the generalized (2+1)-dimensional combined sinh-cosh-Gordon [55] $u_{tt} - u_{xx} - u_{yy} + \delta[\sinh(nu) + \cosh(nu)] = 0$.

Case 5 $p(u) = \sigma + \delta\ln u$, where σ is a constant and δ is nonzero constant.

This case reduces to Case 1.

2.5 Travelling wave solutions of two cases

In order to obtain exact solutions, one has to solve the associated Lagrange's equations

$$\frac{dt}{\tau(t,x,y,u)} = \frac{dx}{\xi(t,x,y,u)} = \frac{dy}{\psi(t,x,y,u)} = \frac{du}{\eta(t,x,y,u)}.$$

We consider two nonlinear cases, namely, Case 3.2 and Case 4.2.

2.5.1 Group-invariant solution of Case 3.2

In this case the equation (2.1) takes the form

$$u_{tt} - u_{xx} - u_{yy} + \delta u^n = 0, \quad n \neq 0, 1. \tag{2.9}$$

We use the Lie point symmetry $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ to reduce equation (2.9) into a PDE with two new independent variables z, w and v as the new dependent variable. The symmetry Γ yields the invariants $u = v(z,w)$, $z = x - t$ and $w = y - t$ which transform (2.9) into the nonlinear second-order PDE

$$2v_{zw} + \delta v^n = 0. \tag{2.10}$$

Equation (2.10) admits the four symmetries

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial w}, \quad X_3 = (n-1)z\frac{\partial}{\partial z} + v\frac{\partial}{\partial v}, \quad X_4 = (n-1)w\frac{\partial}{\partial w} + v\frac{\partial}{\partial v}.$$

The symmetry $X_1 + cX_2$ gives rise to the group-invariant solution $v = f(s)$, where $s = w - cz$ and $f(s)$ satisfies the second-order nonlinear ODE

$$2cf''(s) - \delta f(s)^n = 0. \tag{2.11}$$

Multiplying (2.11) by $f'(s)$ and integrating, we obtain

$$\frac{\delta f(s)^{n+1}}{n+1} - cf'^2(s) = C_1, \tag{2.12}$$

where C_1 is an arbitrary constant of integration. Equation (2.12) is a variables separable equation, which on integration yields

$$-\frac{cf(s)\sqrt{\delta f(s)^{n+1}-C_1(n+1)}}{C_1\sqrt{c(n+1)}}\ {}_2F_1\left(1,\frac{1}{2}+\frac{1}{n+1};1+\frac{1}{n+1};\frac{\delta f(s)^{n+1}}{nC_1+C_1}\right)=\pm s+C_2,$$

where C_2 is an arbitrary constant of integration and ${}_2F_1$ is the generalized hypergeometric function [56]. Reverting back to our original variables we obtain the solution of (2.9) in the form

$$-\frac{cu\sqrt{\delta u^{n+1}-C_1(n+1)}}{C_1\sqrt{c(n+1)}}\ {}_2F_1\left(1,\frac{1}{2}+\frac{1}{n+1};1+\frac{1}{n+1};\frac{\delta u^{n+1}}{nC_1+C_1}\right)$$
$$=\pm\{(c-1)t-cx+y\}+C_2.$$

A special solution of (2.9) can be obtained by taking $C_1=0$ in (2.12). Then the integration of (2.12) with $C_1=0$, yields

$$u(t,x,y)=\left(\frac{2}{n-1}\right)^{\frac{2}{n-1}}\left[\sqrt{\frac{\delta}{c(n+1)}}\{(c-1)t-cx+y\}+C_2\right]^{\frac{2}{1-n}},\quad n\neq\pm1.$$

2.5.2 Group-invariant solution of Case 4.2

For the Case 4.2, the equation (2.1) becomes

$$u_{tt}-u_{xx}-u_{yy}+\delta e^{nu}=0,\quad \delta,n\neq0. \tag{2.13}$$

Again using the symmetry $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3$ and the invariants $u=v(z,w)$, $z=x-t$ and $w=y-t$, the equation (2.13) transforms into the nonlinear PDE

$$2v_{zw}+\delta e^{nv}=0. \tag{2.14}$$

This equation admits the point symmetries

$$X_1=\frac{\partial}{\partial z},\quad X_2=\frac{\partial}{\partial w},\quad X_3=f(z)\frac{\partial}{\partial v},\quad X_4=g(w)\frac{\partial}{\partial v},\quad X_5=w\frac{\partial}{\partial w}-z\frac{\partial}{\partial v}.$$

The symmetry X_1+cX_2 gives rise to the group-invariant solution $v=F(s)$, where c is a non-zero constant, $s=w-cz$ is an invariant of X_1+cX_2 and $F(s)$ satisfies the second-order nonlinear ODE

$$2cF''(s)-\delta e^{nF(s)}=0. \tag{2.15}$$

Integrating this equation twice and reverting back to the original variables, we obtain the solution of equation (2.13) in the form

$$u(t,x,y) = \frac{1}{n}\ln\left[\frac{cC_1}{\delta n}\left\{\tanh^2\left(\frac{1}{2}\sqrt{C_1}\,[C_2 + \{(c-1)t - cx + y\}]\right) - 1\right\}\right], \quad (2.16)$$

where C_1 and C_2 are constants of integration.

2.6 The subclass of nonlinearly self-adjoint equations and conservation Laws

In this section we use Ibragimov theorem to obtain conservations laws for the nonlinearly self-adjoint [57–60] subclass of the (2+1)-dimensional Klein-Gordon equation.

2.6.1 Self-adjoint and nonlinearly self-adjoint equations

In this subsection we will derive nonlinearly self-adjoint equation from equation (2.1). Equation (1.40) yields

$$\begin{aligned} E^* &= \frac{\delta}{\delta u}[v(u_{tt} - u_{xx} - u_{yy} + p(u))] \\ &= v_{tt} - v_{xx} - v_{yy} + p'(u)v. \end{aligned} \quad (2.17)$$

Setting $v = h(x,t,u)$ in (2.17) we get

$$\begin{aligned} &p'(u)h + h_{tt} + 2\,u_t h_{tu} + u_{tt}h_u + u_t^2 h_{uu} - h_{xx} - 2u_x h_{xu} \\ &-u_{xx}h_u - u_x^2 h_{uu} - h_{yy} - 2u_y h_{yu} - u_{yy}h_u - u_y^2 h_{uu}. \end{aligned}$$

We now assume that

$$E^* - \lambda(u_{tt} - u_{xx} - u_{yy} + p(u)) = 0, \quad (2.18)$$

where λ is an undetermined coefficient. Condition (2.18) yields

$$\begin{aligned} &p'(u)h + h_{tt} + 2\,u_t h_{tu} + u_{tt}h_u + u_t^2 h_{uu} - h_{xx} - 2u_x h_{xu} - u_{xx}h_u - u_x^2 h_{uu} \\ &-h_{yy} - 2u_y h_{yu} - u_{yy}h_u - u_y^2 h_{uu} - \lambda u_{tt} + \lambda u_{xx} + \lambda u_{yy} - \lambda p(u). \end{aligned}$$

Comparing the coefficients for the different derivatives of u, we obtain

$$h_u - \lambda = 0,\ h_{uu} = 0,\ h_{tu} = 0,\ h_{xu} = 0,\ h_{yu} = 0,$$
$$p'(u)h + h_{tt} - h_{xx} - h_{yy} - p(u)h_u = 0.$$

Solving the above system, we get

$$p(u) = c_2 u, \quad h = c_1 u + B(t,x,y),$$

where c_1, c_2 are constants and $B(t,x,y)$ satisfy the following condition

$$B_{tt} - B_{xx} - B_{yy} + c_2 B = 0. \tag{2.19}$$

We can now state the following theorem:

Theorem 2.6.1 *Equation* (2.1) *is nonlinearly self-adjoint for a function* $p(u) = c_2 u$ *with*

$$h = c_1 u + B(t,x,y)$$

for any function $B(t,x,y)$ *satisfying condition* (2.19).

2.6.2 Conservation laws

In this subsection we use Theorem 1.7.1 on conservation laws proved in [49] in conjuction with Theorem 2.6.1 to derive the conservation laws of the nonlinearly self-adjoint equation.

We now apply Theorem 1.7.1 to find the conserved vectors for the nonlinearly self-adjoint equation

$$u_{tt} - u_{xx} - u_{yy} + c_2 u = 0. \tag{2.20}$$

This equation has the Lagrangian $\mathcal{L}$ given by

$$\mathcal{L} = \Big(c_1 u + B(t,x,y)\Big)\Big(u_{tt} - u_{xx} - u_{yy} + c_2 u\Big) \tag{2.21}$$

and the eight Lie point symmetries

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = x\partial_t + t\partial_x,\ X_5 = y\partial_t + t\partial_y,$$
$$X_6 = -y\partial_x + x\partial_y,\ X_7 = u\partial_u,\ X_8 = F\,(t,x,y)\,\partial_u,$$

where $F = F(t,x,y)$ satisfies $F_{tt} - F_{xx} - F_{yy} + c_2 F = 0$.

The conserved vectors associated with the above eight symmetries are given by

$$C_1^1 = c_1 u(-u_{xx} - u_{yy} + c_2 u) + B(-u_{xx} - u_{yy} + c_2 u) + c_1 u_t^2 + u_t B_t,$$
$$C_1^2 = -c_1 u_t u_x - u_t B_x + c_1 u u_{tx} + u_{tx} B,$$
$$C_1^3 = -c_1 u_t u_y - u_t B_y + c_1 u u_{ty} + u_{ty} B;$$

$$C_2^1 = c_1 u_t u_x + u_t B_x - c_1 u u_{tx} - u_{tx} B,$$
$$C_2^2 = c_1 u(u_{tt} - u_{yy} + c_2 u) + B(u_{tt} - u_{yy} + c_2 u) - c_1 u_x^2 - u_x B_x,$$
$$C_2^3 = -c_1 u_x u_y - u_x B_y + c_1 u u_{xy} + u_{xy} B;$$

$$C_3^1 = c_1 u_t u_y + u_y B_t - c_1 u u_{ty} - u_{ty} B,$$
$$C_3^2 = -c_1 u_x u_y - u_y B_x + c_1 u u_{xy} + u_{xy} B,$$
$$C_3^3 = c_1 u(u_{tt} - u_{xx} + c_2 u) + B(u_{tt} - u_{xx} + c_2 u) - c_1 u_y^2 - u_y B_y;$$

$$\begin{aligned}
C_4^1 &= c_1 x u(-u_{xx} - u_{yy} + c_2 u) + Bx(-u_{xx} - u_{yy} + c_2 u) + c_1 x u_t^2 + c_1 t u_t u_x + x u_t B_t \\
&\quad + t u_x B_t - c_1 u u_x - u_x B - c_1 t u u_{tx} - t u_{tx} B, \\
C_4^2 &= c_1 t u(u_{tt} - u_{yy} + c_2 u) + Bt(u_{tt} - u_{yy} + c_2 u) - c_1 x u_t u_x - c_1 t u_x^2 - x u_t B_x \\
&\quad - t u_x B_x + c_1 u u_t + u_t B + c_1 x u u_{tx} + x u_{tx} B, \\
C_4^3 &= -c_1 x u_t u_y - c_1 t u_x u_y - x u_t B_y - t u_x B_y + c_1 x u u_{ty} \\
&\quad + c_1 t u u_{xy} + x u_{ty} B + t u_{xy} B;
\end{aligned}$$

$$\begin{aligned}
C_5^1 &= c_1 y u(-u_{xx} - u_{yy} + c_2 u) + By(-u_{xx} - u_{yy} + c_2 u) + c_1 y u_t^2 + c_1 t u_t u_y + y u_t B_t \\
&\quad t u_y B_t - c_1 u u_y - u_y B - c_1 t u u_{yt} - t u_{yt} B, \\
C_5^2 &= -c_1 y u_t u_x - c_1 t u_x u_y - y u_t B_x - t u_y B_x + c_1 y u u_{tx} + y u_{tx} B + c_1 t u u_{xy} + t u_{xy} B \\
C_5^3 &= c_1 t u(u_{tt} - u_{xx} + c_2 u) + Bt(u_{tt} - u_{xx} + c_2 u) - c_1 y u_t u_y - c_1 t u_y^2 - y u_t B_y \\
&\quad - t u_y B_y + c_1 u u_t + u_t B + c_1 y u u_{ty} + y u_{ty} B;
\end{aligned}$$

$$
\begin{aligned}
C_6^1 &= -c_1yu_tu_x + c_1xu_tu_y - yu_xB_t + xu_yB_t + c_1yuu_{tx} - c_1xuu_{ty} + yu_{xy}B - xu_{ty}B,\\
C_6^2 &= -c_1yu(u_{tt} - u_{yy} + c_2u) - By(u_{tt} - u_{yy} + c_2u) + c_1yu_x^2 - c_1xu_xu_y + yu_xB_x\\
&\quad -xu_yB_x + c_1uu_y + u_yB + c_1xuu_{xy} + xu_{xy}B,\\
C_6^3 &= c_1xu(u_{tt} - u_{xx} + c_2u) + Bx(u_{tt} - u_{xx} + c_2u) + c_1yu_xu_y - c_1xu_y^2 + yu_xB_y\\
&\quad -xu_yB_y - c_1uu_x - u_xB - c_1yuu_{xy} - yu_{xy}B;
\end{aligned}
$$

$$
\begin{aligned}
C_7^1 &= u_tB - uB_t,\\
C_7^2 &= uB_x - u_xB\\
C_7^3 &= uB_y - Bu_y;
\end{aligned}
$$

$$
\begin{aligned}
C_8^1 &= c_1uF_t + F_tB - FB_t - c_1u_tF,\\
C_8^2 &= c_1u_xF + FB_x - c_1uF_x - F_xB\\
C_8^3 &= c_1u_yF + FB_y - c_1uF_y - F_yB,
\end{aligned}
$$

respectively, where the functions $B(t, x, y)$ and $F(t, x, y)$ satisfy the equation

$$\phi_{tt} - \phi_{xx} - \phi_{yy} + c_2\phi = 0.$$

2.7 Conclusion

In this chapter Lie group classification was performed on the generalized (2+1)-dimensional Klein-Gordon equation (2.1). The functional forms of the generalized (2+1)-dimensional Klein-Gordon equation of the type linear, power, exponential and logarithmic were obtained. From the classification we retrieved two special equations, namely, the generalized Liouville equation in (2+1) dimension and the (2+1)-dimensional generalized combined sinh-cosh-Gordon equation. In addition, the group-invariant solutions of the generalized (2+1)-dimensional Klein-Gordon equation were derived for power law and exponential function cases. We have also illustrated that the generalized (2+1)-dimensional Klein-Gordon equation is nonlinearly self-adjoint under the conditions given in Theorem 2.6.1. Lastly conservation laws for the nonlinearly self-adjoint subclass were derived by using the new conservation theorem due to Ibragimov.

Chapter 3

Symmetry reductions, exact solutions and conservation laws of a generalized double sinh-Gordon equation

In this chapter, we study a generalized double sinh-Gordon equation, namely

$$u_{tt} - ku_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu) = 0, \qquad n \geq 1, \tag{3.1}$$

where k, α and β are non-zero real constants. The above equation appears in several physical phenomena such as integrable quantum field theory, kink dynamics and fluid dynamics. It should be noted that when $n = k = 1$, $\alpha = 1/2$ and $\beta = 0$, (3.1) reduces to the sinh-Gordon equation [61]. Furthermore, if $k = a$, $\alpha = b/2$ and $\beta = 0$, (3.1) becomes the generalized sinh-Gordon equation [62]. Various methods have been used to study (3.1). In [13] the tanh method and variable separable ODE method was employed to find the exact solutions of (3.1). The authors of [16] studied the existence of periodic wave, solitary wave, kink and anti-kink wave and unbounded wave solutions of (3.1) by using the method of bifurcation theory of dynamical systems. The solitary and periodic wave solutions of (3.1) were obtained in [44] by employing (G'/G)-expansion method. In addition, it was shown that the solutions obtained in [44] were more general than those obtained in [13]. Here we use Lie symmetry analysis together with the exponential-function method and the simplest equation method to obtain exact solutions for this equation. Moreover, we derive conservation laws for the underlying equation by using four different approaches, namely, the direct method, the Noether theorem, the new conservation theorem due to Ibragimov and the multiplier method.

Part of this work has been published in [63]. The other part has been submitted for publication. See [64].

3.1 Symmetry reductions and exact solutions of (3.1)

We assume that the vector field of the form

$$X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

will generate the symmetry group of (3.1). Applying the second prolongation $X^{[2]}$ to (3.1) we obtain an overdetermined system of eight linear partial differential equations, namely

$$\begin{aligned}
&\xi_u = 0, \tau_u = 0, \eta_{uu} = 0, \xi_t - k\tau_x = 0,\\
&\tau_t - \xi_x = 0, \tau_{tt} - k\tau_{xx} - 2\eta_{tu} = 0, \xi_{tt} - k\xi_{xx} + 2k\eta_{xu} = 0,\\
&-2\beta n\,\eta + 4\alpha\,\tau_t \sinh(nu) - 2\alpha\,\eta_u \sinh(nu) + 2\alpha n\,\eta \cosh(nu) + 4\beta n\,\eta \cosh^2(nu)\\
&+4\beta\,\tau_t \cosh(nu)\sinh(nu) - 2\beta\,\eta_u \cosh(nu)\sinh(nu) + \eta_{tt} - k\eta_{xx} = 0.
\end{aligned}$$

Solving the above equations one obtains the following three Lie point symmetries:

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = kt\partial_x + x\partial_t.$$

3.1.1 One-dimensional optimal system of subalgebras

In this subsection we first obtain the optimal system of one-dimensional subalgebras of (3.1). Thereafter the optimal system will be used to obtain the optimal system of group-invariant solutions of (3.1). For this purpose we invoke the method given in [23]. Recall that the commutator of X_i and X_j, denoted by $[X_i, X_j]$, is given by

$$[X_i, X_j] = X_i X_j - X_j X_i$$

and the adjoint transformations are given by

$$\mathrm{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2}\epsilon^2[X_i, [X_i, X_j]] - \cdots .$$

The commutator table of the Lie point symmetries of (3.1) and the adjoint representations of the symmetry group of (3.1) on its Lie algebra are presented in Table 1 and Table 2, respectively.

Table 1. Commutator table of the Lie algebra of system (3.1)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	X_2
X_2	0	0	kX_1
X_3	$-X_2$	$-kX_1$	0

Table 2. Adjoint table of the Lie algebra of system (3.1)

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \epsilon X_2$
X_2	X_1	X_2	$X_3 - k\epsilon X_1$
X_3	$\cosh(\sqrt{k}\epsilon)X_1 + \frac{1}{\sqrt{k}}\sinh(\sqrt{k}\epsilon)X_2$	$\sqrt{k}\sinh(\sqrt{k}\epsilon)X_1 + \cosh(\sqrt{k}\epsilon)X_2$	X_3

Thus, from Tables 1 and 2 and following the method given in [23] one can conclude that an optimal system of one-dimensional subalgebras of (3.1) is given by $\{cX_1 + X_2, X_2, X_3\}$, where c is a non-zero constant.

3.1.2 Symmetry reductions of (3.1)

Here the optimal system of one-dimensional subalgebras constructed above will be used to obtain symmetry reductions. Thereafter, we will obtain the exact solutions of (3.1).

Case 1. $cX_1 + X_2$

The symmetry generator $cX_1 + X_2$ gives rise to the group-invariant solution

$$u = W(z), \tag{3.2}$$

where $z = x - ct$ is an invariant of the symmetry $cX_1 + X_2$ and W is an arbitrary function of z. The insertion of (3.2) into (3.1) yields the ODE

$$(c^2 - k)W''(z) + 2\alpha \sinh(nW(z)) + \beta \sinh(2nW(z)) = 0. \tag{3.3}$$

Using the transformation

$$W(z) = \frac{1}{n}\ln(H(z)) \tag{3.4}$$

on (3.3) we obtain the nonlinear second-order ordinary differential equation

$$\begin{aligned}&2(c^2 - k)H(z)H''(z) - 2(c^2 - k)H'(z)^2 + 2\alpha nH(z)^3 - 2\alpha nH(z)\\&+\beta nH(z)^4 - \beta n = 0.\end{aligned} \tag{3.5}$$

The integration of the above equation and reverting back to original variables, yields

$$\pm \int \left[\frac{1}{k - c^2}\left(2\alpha n \exp(nu) + 2\alpha n \exp(3nu) + \frac{1}{2}\beta n + \frac{1}{2}\beta n \exp(4nu)\right) + c_1 \exp(2nu)\right]^{-\frac{1}{2}} \times n\exp(nu)du = x - ct + c_2,$$

where c_1 and c_2 are constants of integration.

Case 2. X_2

The symmetry operator X_2 results in the group-invariant solution of the form

$$u = W(z), \tag{3.6}$$

where $z = x$ is an invariant of X_2 and W is an arbitrary function satisfying the ODE

$$-kW''(z) + 2\alpha \sinh(nW(z)) + \beta \sinh(2nW(z)) = 0. \tag{3.7}$$

Again using the transformation (3.4), equation (3.7) becomes

$$-2kH(z)H''(z) + 2kH'(z)^2 + 2\alpha nH(z)^3 - 2\alpha nH(z) + \beta nH(z)^4 - \beta n = 0, \tag{3.8}$$

whose solution is

$$\pm \int \left[\frac{1}{k}\left(2\alpha n \exp(nu) + 2\alpha n \exp(3nu) + \frac{1}{2}\beta n + \frac{1}{2}\beta n \exp(4nu)\right) + c_1 \exp(2nu)\right]^{-\frac{1}{2}} \times n \exp(nu) du = x + c_2,$$

where c_1 and c_2 are constants of integration and we obtain a steady state solution.

Case 3. X_3

The symmetry X_3, gives rise to the group-invariant solution

$$u = W(z), \tag{3.9}$$

where $z = x^2 - kt^2$ is an invariant of X_3 and W satisfies the ODE

$$4kzW''(z) - 2kW'(z) + 2\alpha \sinh(nW(z)) + \beta \sinh(2nW(z)) = 0.$$

3.1.3 Exact solutions of (3.1) using exponential-function method

In this subsection we employ the exponential-function method to solve equation (3.5). This method was introduced by He and Wu [9]. The exponential-function method results in the travelling wave solution based on the assumption that the solution of (3.5) can be expressed in the form

$$H(z) = \frac{\sum_{n=-c}^{d} a_n \exp(nz)}{\sum_{m=-p}^{q} b_m \exp(mz)}, \tag{3.10}$$

where c, d, p and q are positive integers that can be determined, and a_n and b_m are unknown constants.

We assume that the solution of (3.5) can be expressed as

$$H(z) = \frac{a_c \exp(cz) + ... + a_{-d} \exp(-dz)}{b_p \exp(pz) + ... + b_{-q} \exp(-qz)}. \tag{3.11}$$

The values of c and d, p and q can be determined by balancing the linear term of the highest order with the highest order of nonlinear term in (3.5), i.e., HH'' and H^4. By straight forward calculation, we have

$$HH'' = \frac{c_1 \exp[(2c + 3p)z] + ...}{c_2 \exp[5pz] + ...} \tag{3.12}$$

and

$$H^4 = \frac{c_3 \exp[4cz] + ...}{c_4 \exp[4pz] + ...} = \frac{c_3 \exp[(4c+p)z] + ...}{c_4 \exp[5pz] + ...}, \tag{3.13}$$

where c_i are coefficients only for simplicity. Balancing the highest order of exponential-function in (3.12) and (3.13), we have $2c+3p = 4c+p$, which yields $c = p$. Similarly, we balance the lowest order in (3.5) to determine values of d and q. We have

$$HH'' = \frac{... + s_1 \exp[-(2d+3q)z]}{... + s_2 \exp[-5qz]} \tag{3.14}$$

and

$$H^4 = \frac{... + s_3 \exp[4dz]}{... + s_4 \exp[-4qz]} = \frac{... + s_3 \exp[-(4d+q)z]}{... + s_4 \exp[-5qz]}, \tag{3.15}$$

where s_i are coefficients only for simplicity. Balancing the lowest order of exponential-function in (3.14) and (3.15), we have $2d + 3q = 4d + q$, which yields $d = q$. For simplicity, we first set $c = p = 1$ and $d = q = 1$, then (3.11) reduces to

$$H(z) = \frac{a_1 \exp(z) + a_0 + a_{-1} \exp(-z)}{b_1 \exp(z) + b_0 + b_{-1} \exp(-z)}. \tag{3.16}$$

Inserting (3.16) into (3.5) and using Maple, we obtain

$$\begin{aligned}&\frac{1}{B}[C_4 \exp(4z) + C_3 \exp(3z) + C_2 \exp(2z) + C_1 \exp(z) + C_0 + C_{-1} \exp(-z)\\ &+C_{-2} \exp(-2z) + C_{-3} \exp(-3z) + C_{-4} \exp(-4z)] = 0,\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
B &= (b_1 \exp(z) + b_0 + b_{-1} \exp(-z))^4,\\
C_4 &= 2\alpha a_1^3 b_1 n - \beta b_1^4 n + \beta a_1^4 n - 2\alpha a_1 b_1^3 n,\\
C_3 &= -2a_1^2 b_0 b_1 c^2 + 2a_1 a_0 b_1^2 c^2 + 6\alpha a_0 a_1^2 b_1 n - 6\alpha a_1 b_0 b_1^2 n + 2a_1^2 b_0 b_1 k - 2a_0 a_1 b_1^2 k\\
&\quad +2\alpha a_1^3 b_0 n - 2a_0 a_1 b_1^2 k + 2\alpha a_1^3 b_0 n + 4\beta a_0 a_1^3 n - 2\alpha a_0 b_1^3 n - 4\beta b_0 b_1^3 n,\\
C_2 &= 4\beta a_{-1} a_1^3 n - 8a_1^2 b_{-1} b_1 c^2 + 8a_{-1} a_1 b_1^2 c^2 + 8a_1^2 b_{-1} b_1 k - 8a_{-1} a_1 b_1^2 k + 2\alpha a_1^3 b_{-1} n\\
&\quad -2\alpha a_{-1} b_1^3 n - 4\beta b_{-1} b_1^3 n + 6\alpha a_0 a_1^2 b_0 n + 6\alpha a_0^2 a_1 b_1 n - 6\alpha a_1 b_0^2 b_1 n - 6\beta b_0^2 b_1^2 n\\
&\quad +6\alpha a_{-1} a_1^2 b_1 n - 6\alpha a_1 b_1^2 b_{-1} n + 6\beta a_0^2 a_1^2 n - 6\alpha a_0 b_0 b_1^2 n,\\
C_1 &= -2a_0^2 b_0 b_1 c^2 + 2a_0 a_1 b_0^2 c^2 + 2a_0^2 b_0 b_1 k - 2a_0 a_1 b_0^2 k - 2a_1^2 b_0 b_{-1} c^2 + 2a_{-1} a_0 b_1^2 c^2
\end{aligned}$$

$$
\begin{aligned}
& -2a_0a_{-1}b_1^2k + 2\alpha a_0^3b_1n + 4\beta a_0^3a_1n - 2\alpha a_1b_0^3n - 4\beta b_0^3b_1n + 12a_{-1}a_1b_0b_1c^2 \\
& -12a_{-1}a_1b_0b_1k + 12a_0a_1b_{-1}b_1k + 6\alpha a_0^2a_1b_0n - 6\alpha a_0b_0^2b_1n + 12\alpha a_{-1}a_0a_1b_1n \\
& -12\alpha a_1b_{-1}b_0b_1n + 6\alpha a_{-1}a_1^2b_0n - 6\alpha a_0b_{-1}b_1^2n - 6\alpha a_{-1}b_0b_1^2n + 6\alpha a_0a_1^2b_{-1}n \\
& +12\beta a_{-1}a_0a_1^2n - 12\beta b_{-1}b_0b_1^2n + 2a_1^2b_0b_{-1}k - 12a_0a_1b_{-1}b_1c^2, \\
C_0 &= 2\alpha a_0^3b_0n - 2\alpha a_0b_0^3n + \beta a_0^4n + 6\alpha a_{-1}a_1^2b_{-1}n + 6\alpha a_0^2a_1b_{-1}n + 6\alpha a_{-1}^2a_1b_1n \\
& +6\alpha a_{-1}a_0^2b_1n + 12\beta a_{-1}a_0^2a_1n - 6\alpha a_1b_{-1}^2b_1n - 6\alpha a_1b_{-1}b_0^2n - 6\alpha a_{-1}b_{-1}b_1^2n \\
& -\beta b_0^4n - 6\alpha a_{-1}b_0^2b_1n - 12\beta b_{-1}b_0^2b_1n + 8a_{-1}a_1b_0^2c^2 - 8a_0^2b_{-1}b_1c^2 - 8a_{-1}a_1b_0^2k \\
& +8a_0^2b_{-1}b_1k + 6\beta a_{-1}^2a_1^2n - 6\beta b_{-1}^2b_1^2n + 12\alpha a_{-1}a_0a_1b_0n - 12\alpha a_0b_{-1}b_0b_1n, \\
C_{-1} &= 12\alpha a_{-1}a_0a_1b_{-1}n - 12\alpha a_{-1}b_{-1}b_0b_1n + 2a_{-1}a_0b_0^2c^2 - 2a_0^2b_{-1}b_0c^2 + 2a_0^2b_{-1}b_0k \\
& -2a_{-1}a_0b_0^2k + 2a_0a_1b_{-1}^2c^2 - 2a_{-1}^2b_0b_1c^2 - 2a_0a_1b_{-1}^2k + 2a_{-1}^2b_0b_1k + 2\alpha a_0^3b_{-1}n \\
& +4\beta a_{-1}a_0^3n - 2\alpha a_{-1}b_0^3n - 4\beta b_{-1}b_0^3n + 12a_{-1}a_1b_{-1}b_0c^2 - 12a_{-1}a_0b_{-1}b_1c^2 \\
& -12a_{-1}a_1b_{-1}b_0k + 12a_{-1}a_0b_{-1}b_1k + 6\alpha a_{-1}a_0^2b_0n - 6\alpha a_0b_{-1}b_0^2n + 6\alpha a_{-1}^2a_1b_0n \\
& +6\alpha a_{-1}^2a_0b_1n + 12\beta a_{-1}^2a_0a_1n - 6\alpha a_1b_{-1}^2b_0n - 6\alpha a_0b_{-1}^2b_1n - 12\beta b_{-1}^2b_0b_1n, \\
C_{-2} &= 2\alpha a_{-1}^3b_1n + 8a_{-1}a_1b_{-1}^2c^2 + 8a_{-1}^2b_{-1}b_1k - 8a_{-1}a_1b_{-1}^2k + 4\beta a_{-1}^3a_1n - 4\beta b_{-1}^3b_1n \\
& -2\alpha a_1b_{-1}^3n - 8a_{-1}^2b_{-1}b_1c^2 + 6\alpha a_{-1}a_0^2b_{-1}n + 6\alpha a_{-1}^2a_0b_0n - 6\alpha a_0b_{-1}^2b_0n \\
& +6\alpha a_{-1}^2a_1b_{-1}n - 6\alpha a_{-1}b_{-1}^2b_1n + 6\beta a_{-1}^2a_0^2n - 6\beta b_{-1}^2b_0^2n - 6\alpha a_{-1}b_{-1}b_0^2n, \\
C_{-3} &= 6\alpha a_0a_{-1}^2b_{-1}n - 6\alpha a_{-1}b_{-1}^2b_0n - 2a_{-1}^2b_{-1}b_0c^2 + 2a_{-1}a_0b_{-1}^2c^2 + 2a_{-1}^2b_0b_{-1}k \\
& -2a_{-1}a_0b_{-1}^2k + 2\alpha a_{-1}^3b_0n + 4\beta a_0a_{-1}^3n - 2\alpha a_0b_{-1}^3n - 4\beta b_0b_{-1}^3n, \\
C_{-4} &= \beta a_{-1}^4n - \beta b_{-1}^4n + 2\alpha a_{-1}^3b_{-1}n - 2\alpha a_{-1}b_{-1}^3n.
\end{aligned}
$$

Equating the coefficients of $\exp(z)$ in (3.17) to zero, we obtain a set of algebraic equations

$$
\begin{aligned}
& C_4 = 0, \quad C_3 = 0, \quad C_2 = 0, \quad C_1 = 0, \quad C_0 = 0, \\
& C_{-1} = 0, \quad C_{-2} = 0, \quad C_{-3} = 0, \quad C_{-4} = 0. \qquad (3.18)
\end{aligned}
$$

Solving the system (3.18) with the help of Maple, we obtain the following three cases:

Case 1

$$
a_{-1} = b_{-1}, \quad a_0 = -b_0, \quad a_1 = b_1, \quad \beta = \frac{\alpha b_0^2 - 4\alpha b_1 b_{-1}}{4b_1 b_{-1}},
$$

$$k = \frac{\alpha b_0^2 n + 2b_{-1}b_1c^2}{2b_{-1}b_1}. \tag{3.19}$$

Case 2

$$a_{-1} = \frac{b_{-1}b_1}{a_1}, \quad a_0 = 0, \quad b_0 = 0, \quad \alpha = \frac{-\beta(a_1^2 + b_1^2)}{2a_1b_1},$$
$$k = \frac{-2\beta a_1^2b_1^2n + \beta a_1^4 n + \beta b_1^4 n + 8a_1^2b_1^2c^2}{8a_1^2b_1^2}. \tag{3.20}$$

Case 3

$$a_{-1} = -\phi b_1, \quad b_{-1} = -\phi a_1, \quad \alpha = \frac{-\beta(a_1^2 + b_1^2)}{2a_1b_1},$$
$$k = \frac{-2\beta a_1^2b_1^2n + \beta a_1^4 n + \beta b_1^4 n + 2a_1^2b_1^2c^2}{2a_1^2b_1^2}, \tag{3.21}$$

where $\phi = \dfrac{-a_0a_1^2b_0 + a_0^2a_1b_1 + a_1b_0^2b_1 - a_0b_0b_1^2}{(a_1 - b_1)^2(a_1 + b_1)^2}$.

Substituting values from (3.19) into (3.16), we obtain

$$H(z) = \frac{b_1\exp(z) - b_0 + b_{-1}\exp(-z)}{b_1\exp(z) + b_0 + b_{-1}\exp(-z)}.$$

As a result one of the solutions of (3.1) is given by

$$u_1(x,t) = \frac{1}{n}\ln\left(\frac{b_1\exp(z) - b_0 + b_{-1}\exp(-z)}{b_1\exp(z) + b_0 + b_{-1}\exp(-z)}\right), \tag{3.22}$$

where $z = x - ct$, $\beta = \dfrac{\alpha b_0^2 - 4\alpha b_1b_{-1}}{4b_1b_{-1}}$ and $k = \dfrac{\alpha b_0^2 n + 2b_{-1}b_1c^2}{2b_1b_{-1}}$.

As a special case, if we choose $b_0 = 2$ and $b_{-1} = b_1 = 1$ in (3.22), then we get $\beta = 0$, $k = 2\alpha n + c^2$ and obtain the solution of the generalized sinh-Gordon equation as

$$u_1(x,t) = \frac{1}{n}\ln(\tanh^2[(1/2)(x - ct)]), \tag{3.23}$$

which is the solution obtained in [12,45].

Now substituting the values from (3.20) (Case 2) into (3.16) results in the second solution of (3.1) as

$$u_2(x,t) = \frac{1}{n}\ln\left(\frac{a_1\exp(z) + \frac{b_{-1}b_1}{a_1}\exp(-z)}{b_1\exp(z) + b_{-1}\exp(-z)}\right) \tag{3.24}$$

with $z = x - ct$, $\alpha = \dfrac{-\beta(a_1^2 + b_1^2)}{2a_1b_1}$, $k = \dfrac{-2\beta a_1^2b_1^2n + \beta a_1^4 n + \beta b_1^4 n + 8a_1^2b_1^2c^2}{8a_1^2b_1^2}$.

The third solution of (3.1) is obtained by using the values from (3.21) (Case 3) and substituting them into (3.16). Consequently, it is given by

$$u_3(x,t) = \frac{1}{n}\ln\left(\frac{a_1\exp(z) + a_0 - b_1\phi\exp(-z)}{b_1\exp(z) + b_0 - a_{-1}\phi\exp(-z)}\right), \tag{3.25}$$

where $z = x - ct$, $\phi = \dfrac{-a_0a_1^2b_0 + a_0^2a_1b_1 + a_1b_0^2b_1 - a_0b_0b_1^2}{(a_1-b_1)^2(a_1+b_1)^2}$, $\alpha = \dfrac{-\beta(a_1^2+b_1^2)}{2a_1b_1}$ and $k = \dfrac{-2\beta a_1^2b_1^2n + \beta a_1^4n + \beta b_1^4n + 2a_1^2b_1^2c^2}{2a_1^2b_1^2}$.

To construct more solutions of (3.1), we now set $c = p = 2$ and $d = q = 2$. Then (3.11) reduces to

$$H(z) = \frac{a_2\exp(2z) + a_1\exp(z) + a_0 + a_{-1}\exp(-z) + a_{-2}\exp(-2z)}{b_2\exp(z) + b_1\exp(z) + b_0 + b_{-1}\exp(-z) + b_{-2}\exp(-2z)}. \tag{3.26}$$

Proceeding as above, we obtain the following three solutions of (3.1):

$$u_4(x,t) = \frac{1}{n}\ln\left(\frac{a_2\exp(2z) + \frac{a_{-1}b_1}{b_{-1}}\exp(z) + \frac{a_{-1}b_0}{b_{-1}} + a_{-1}\exp(-z)}{\frac{a_2b_{-1}}{a_{-1}}\exp(z) + b_1\exp(z) + b_0 + b_{-1}\exp(-z)}\right), \tag{3.27}$$

where $z = x - ct$, $\alpha = -\dfrac{\beta\left(a_{-1}^2 + b_{-1}^2\right)}{2a_{-1}b_{-1}}$;

$$u_5(x,t) = \frac{1}{n}\ln\left(\frac{a_2\exp(2z) + a_1\exp(z) + b_0}{-a_2\exp(z) + b_1\exp(z) + b_0}\right), \tag{3.28}$$

with $z = x - ct$, $\beta = \dfrac{\alpha\left(b_1^2 + 4a_2b_0\right)}{4a_2b_0}$, $k = \dfrac{\alpha nb_1^2 + 2a_2b_0c^2}{2a_2b_0}$;

and

$$u_6(x,t) = \frac{1}{n}\ln\left(\frac{a_2\exp(2z) - b_0 + b_{-2}\exp(-2z)}{a_2\exp(2z) + b_0 + b_{-2}\exp(-2z)}\right), \tag{3.29}$$

where $z = x - ct$, $\alpha = -\dfrac{8a_2b_{-2}\left(c^2 - k\right)}{b_0^2n}$, $\beta = \dfrac{2\left(4a_2b_{-2}c^2 - 4a_2b_{-2}k - b_0^2c^2 + b_0^2k\right)}{b_0^2n}$.

By taking $n = 2$, $b_{-1} = -1$, $b_0 = 2$, $c = 1$ and $b_1 = -1$ in the solution (3.22), we have its profile given in Figure 3.1.

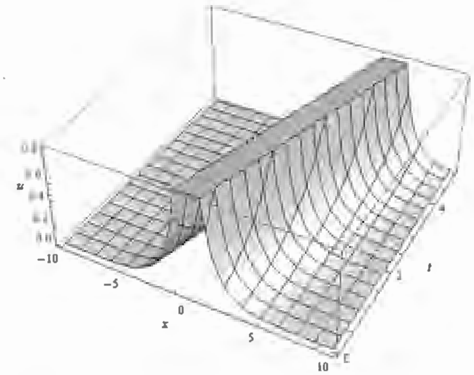

Figure 3.1: Profile of solution (3.22)

By taking $n = 3$, $b_{-2} = 1$, $b_0 = 2$, $c = 1$ and $a_1 = 1$ in the solution (3.29), we have its profile given in Figure 3.2.

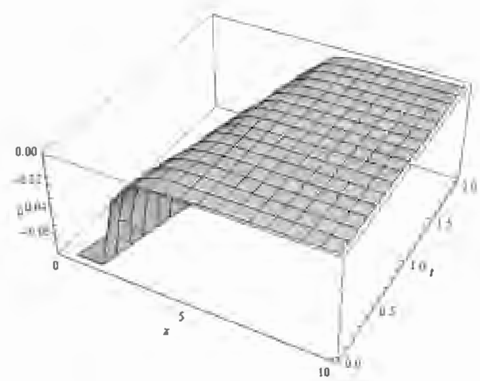

Figure 3.2: Profile of solution (3.29)

3.1.4 Exact solutions using simplest equation method

In this subsection we invoke the simplest equation method to solve the highly nonlinear ODE (3.5). This method was introduced by Kudryashov [65, 66] and later modified by Vitanov [67]. This will then give us the exact solution for the generalized double sinh-Gordon equation (3.1). The simplest equations that we will use are the Bernoulli and Riccati equations.

Here we first present the simplest equation method and consider the solutions of

(3.5) in the form

$$H(z) = \sum_{i=0}^{M} A_i (G(z))^i, \tag{3.30}$$

where $G(z)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure as in [67] and $A_0, \cdots, A_M$ are parameters to be determined. We note that the Bernoulli and Riccati equations are well-known nonlinear ODEs whose solutions can be expressed in terms of elementary functions. Let us consider here the Bernoulli equation

$$G'(z) = aG(z) + bG^2(z), \tag{3.31}$$

where a and b are arbitrary constants. The general solution to (3.31) is given by

$$G(z) = a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}.$$

In case of the Riccati equation

$$G'(z) = aG^2(z) + bG(z) + \nu, \tag{3.32}$$

where a, b and ν are arbitrary constants, we shall use the solutions

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]$$

and

$$G(z) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)},$$

with $\theta^2 = b^2 - 4a\nu > 0$ and C is an arbitrary constant of integration.

Solutions of (3.1) using the Bernoulli equation as the simplest equation

The balancing procedure [67] yields $M = 1$, so the solutions of (3.5) take the form

$$H(z) = A_0 + A_1 G. \tag{3.33}$$

Inserting (3.33) into (3.5) and using the Bernoulli equation (3.31) and thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of five equations in terms of A_0, A_1, namely

$$
\begin{aligned}
\beta n A_0^4 - \beta n + 2\alpha n A_0^3 - 2\alpha n A_0 &= 0,\\
4\beta n A_0^3 A_1 - 2a^2 A_0 A_1 k - 2\alpha n A_1 + 6\alpha n A_0^2 A_1 + 2a^2 A_0 A_1 c^2 &= 0,\\
\alpha n A_0 A_1^2 - a A_0 A_1 bk + \beta n A_0^2 A_1^2 + A_0 A_1 abc^2 &= 0,\\
\beta n A_1^2 + 2A_1^2 b^2 c^2 - 2A_1^2 b^2 k &= 0,\\
+4\beta A_0 A_1^3 n + 4A_0 A_1 b^2 c^2 - 4A_0 A_1 b^2 k + 2\alpha n A_1^3 + 2A_1^2 abc^2 - 2kA_1^2 ab &= 0.
\end{aligned}
$$

Solving the above system of algebraic equations, with the aid of Maple, one possible solution is

$$A_1 = \frac{b(A_0^2 - 1)}{aA_0},\ \alpha = \frac{\beta(A_0^2 + 1)}{2A_0},\ \beta = -\frac{2b^2(c^2 - k)}{nA_1^2}.$$

Thus, reverting back to the original variables, a solution of (3.1) is

$$u(t,x) = \frac{1}{n}\ln\left(A_0 + A_1 a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\right), \qquad (3.34)$$

where $z = x - ct$ and C is an arbitrary constant of integration.

Solutions of (3.1) using the Riccati equation as the simplest equation

In this case the balancing procedure yields $M = 1$. So the solutions of (3.5) take the form

$$H(z) = A_0 + A_1 G. \qquad (3.35)$$

Substituting (3.35) into (3.5) and making use of the Riccati equation (3.32), we obtain an algebraic system of equations in terms of A_0, A_1 by equating the coefficients powers of G^i to zero. The resulting algebraic equations are

$$-2A_0 A_1 bk\nu + 2\alpha n A_0^3 - \beta n + 2A_1^2 k\nu^2 - 2A_1^2 c^2\nu^2 - 2\alpha n A_0 + \beta n A_0^4 + 2A_0 A_1 bc^2\nu = 0,$$

$$-2A_1^2 a^2 k + 2a^2 A_1^2 c^2 + \beta n A_1^4 = 0,$$

$$-4aA_0A_1k\nu + 4\beta nA_0^3A_1 + 2A_1^2bk\nu + 4aA_0A_1c^2\nu - 2A_1^2bc^2\nu - 2A_0A_1b^2k + 6\alpha nA_0^2A_1$$
$$-2\alpha nA_1 + 2A_0A_1b^2c^2 = 0,$$
$$-abk + \beta nA_0A_1 + abc^2 + \alpha nA_1 = 0,$$
$$2a^2A_0A_1c^2 + aA_1^2bc^2 - aA_1^2bk - 2a^2A_0A_1k + \alpha nA_1^3 + 2\beta nA_0A_1^3 = 0.$$

Solving the above equations, we get

$$A_0 = \frac{A_1b + \sqrt{A_1^2b^2 + 4a^2 - 4A_1^2a\nu}}{2a},$$
$$\alpha = \frac{-\beta(A_0^2a + a - A_1^2\nu)}{2aA_0},$$
$$\beta = \frac{-2a(c^2 - k)}{nA_1^2},$$

and consequently, the solutions of (3.1) are

$$u(t,x) = \frac{1}{n}\ln\left(A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z + C)\right]\right\}\right) \tag{3.36}$$

and

$$u(t,x) = \frac{1}{n}\ln\left(A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}\right), \tag{3.37}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

3.2 Conservation laws of (3.1)

In this section conservation laws will be constructed for (3.1) by using four different methods, namely, the direct method, the Noether theorem, the new conservation theorem and the multiplier method.

We recall that the equation (3.1) admits the following three Lie point symmetry generators:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = x\partial_t + kt\partial_x.$$

3.2.1 Application of the direct method

It is well-known that there exists a fundamental relationship between the point symmetries admitted by a given equation and the conservation laws of that equation. Following [39], we see that the conservation law

$$D_tT^1 + D_xT^2 = 0, \tag{3.38}$$

which must be evaluated on the partial differential equation, can be considered together with the following requirements:

$$X^{[n]}(T^1) + T^1 D_x(\xi) - T^2 D_x(\tau) = 0, \tag{3.39}$$

$$X^{[n]}(T^2) + T^2 D_t(\tau) - T^1 D_t(\xi) = 0 \tag{3.40}$$

in which $X^{[n]}$ is the *nth* prolongation of a point symmetry of the original equation. The order of the extension equals to the order of the highest derivative in T^1 and T^2. Consequently, for given X, (3.38)-(3.40) can be solved to obtain the conserved vectors or tuple $T = (T^1, T^2)$.

The condition (3.38) on the equation (3.1) gives

$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + \Big(ku_{xx} - 2\alpha\sinh nu - \beta\sinh(2nu)\Big)\frac{\partial T^1}{\partial u_t} + u_{tx}\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x}$$
$$+u_x\frac{\partial T^2}{\partial u} + u_{tx}\frac{\partial T^2}{\partial u_t} + u_{xx}\frac{\partial T^2}{\partial u_x} = 0.$$

Since T^1 and T^2 are independent of the second derivatives of u, it implies that the coefficients of u_{tt}, u_{tx} and u_{xx} must be zero. Hence,

$$\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial u_t} = 0, \tag{3.41}$$

$$k\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial u_x} = 0, \tag{3.42}$$

$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} - \Big(2\alpha\sinh(nu) + \beta\sinh(2nu)\Big)\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} = 0. \tag{3.43}$$

We now construct the conservation laws for (3.1) using the three admitted Lie point symmetries.

We start with the translation symmetry $X_1 = \partial_t$, which is already in its extended form. The symmetry conditions (3.39)-(3.40) yield

$$\frac{\partial T^1}{\partial t} = 0, \quad \frac{\partial T^2}{\partial t} = 0, \tag{3.44}$$

respectively. Therefore from (3.41)-(3.43) and (3.44) the components of the conserved vector of (3.1) associated with the symmetry X_1 are given by

$$\begin{aligned} T^1 &= \frac{c_4 u_t^2}{k} + c_4 u_x^2 + c_5 u_x + \frac{2c_4}{k}\left[\frac{2\alpha\cosh(nu)}{n} + \frac{\beta\cosh(2nu)}{2n}\right] + j(x) + c_6, \\ T^2 &= -2c_4 u_t u_x - c_5 u_t + c_7, \end{aligned}$$

where c_4, c_5, c_6 and c_7 are arbitrary constants and $j(x)$ is an arbitrary function of x.

Continuing in the same manner using X_2 and X_3 we obtain the components of the conserved vector for equation (3.1) as

$$\begin{aligned} T^1 &= \frac{c_4 u_t^2}{k} + c_4 u_x^2 + c_5 u_x + \frac{2c_4}{k}\left[\frac{2\alpha\cosh(nu)}{n} + \frac{\beta\cosh(2nu)}{2n}\right] + c_8 \\ T^2 &= -2c_4 u_t u_x - c_5 u_t + p(t), \end{aligned}$$

and

$$\begin{aligned} T^1 &= c_5 u_x + c_6 x, \\ T^2 &= -c_5 u_t + c_6 k t, \end{aligned}$$

respectively, where c_4, c_5, c_6 and c_8 are constants and $p(t)$ is an arbitrary function of t. However, we note that the symmetry X_3 gives a trivial conserved vector.

3.2.2 Application of the Noether theorem

Here we apply the Noether theorem to construct conservation laws of the generalized double sinh-Gordon equation (3.1). It can be easily verified that equation (3.1) has a first-order Lagrangian given by

$$L = \frac{1}{2}u_t^2 - \frac{k}{2}u_x^2 - \left(\frac{2\alpha\cosh(nu)}{n} + \frac{\beta\cosh(2nu)}{2n}\right). \tag{3.45}$$

The insertion of (3.45) into the Noether operator determining equation (1.37) gives

$$-2\alpha\eta\sinh(nu) - \beta\eta\sinh(2nu) + u_t\{\eta_t + u_t\eta_u - u_t(\tau_t + u_t\tau_u) - u_x(\xi_t + u_t\xi_u)\}$$

$$+\{(1/2)u_t^2 - (k/2)u_x^2 - (2\alpha/n)\cosh(nu) - (\beta/2n)\cosh(nu)\}\{\tau_t + u_t\tau_u + \xi_x + u_x\xi_u\}$$
$$-ku_x\{\eta_x + u_x\eta_u - u_t(\tau_x + u_x\tau_u) - u_x(\xi_x + u_x\xi_u)\} = B_t^1 + u_tB_u^1 + B_x^2 + u_xB_u^2.$$

The separation of the above equation on the derivatives of u, yields

$$\tau_u = 0,\ \xi_u = 0,\ \xi_t - k\tau_x = 0,\ 2\eta_u - \tau_t + \xi_x = 0,\ \xi_x - \tau_t - 2\eta_u = 0,$$
$$(k/2)\xi_x - k\eta_u - (k/2)\tau_t = 0,\ \eta_t = B_u^1,\ k\eta_x = -B_u^2,$$
$$2\alpha\sinh(nu)\eta + \beta\sinh(2nu)\eta + 2(\alpha/n)\cosh(nu)\tau_t + 2(\alpha/n)\cosh(nu)\xi_x$$
$$+(\beta/2n)\cosh(2nu)\tau_t + (\beta/2n)\cosh(2nu)\xi_x = -B_t^1 - B_x^2.$$

After some straightforward but lengthy calculations, we obtain

$$\tau = d_1,\ \xi = -d_2,\ \eta = 0,$$
$$B_t^1(t,x) + B_x^2(t,x) = 0,$$

where d_1 and d_2 are arbitrary constants and $B^1(t,x)$ and $B^2(t,x)$ are arbitrary functions of t and x. We can choose $B^1(t,x) = B^2(t,x) = 0$ as they contribute to the trivial part of the conserved vectors. Thus, we get two Noether point symmetries, namely

$$X_1 = \partial_t, \quad X_2 = \partial_x.$$

The use of the theorem due to Noether, with $X_1 = \partial_t$, gives the conserved vector

$$T^1 = -\frac{1}{2}u_t^2 - \frac{k}{2}u_x^2 - \frac{2\alpha}{n}\cosh(nu) - \frac{\beta}{2n}\cosh(2nu), \quad T^2 = ku_tu_x.$$

Using $X_2 = \partial_x$ and employing the Noether theorem, we obtain

$$T^1 = -u_tu_x, \quad T^2 = \frac{1}{2}u_t^2 + \frac{k}{2}u_x^2 - \frac{2\alpha}{n}\cosh(nu) - \frac{\beta}{2n}\cosh(2nu).$$

3.2.3 Application of the new conservation theorem

In this subsection we use the new conservation theorem given in [49] and construct conservation laws for (3.1). The adjoint equation of (3.1), by invoking (1.40), is

$$E^*(t,x,u,v,\ldots,u_{xx},v_{xx}) = \frac{\delta}{\delta u}\Big[v(u_{tt} - ku_{xx} + 2\alpha\sinh(nu) + \beta\sinh(2nu)\Big] = 0, \tag{3.46}$$

where $v = v(t, x)$ is a new dependent variable. Thus from (3.46) we have

$$v_{tt} - kv_{xx} + 2nv\big[\alpha \cosh(nu) + \beta \cosh(2nu)\big] = 0. \qquad (3.47)$$

It is clear from the adjoint equation (3.47) that equation (3.1) is not self-adjoint. By recalling (1.45), we obtain the Lagrangian for the system of equations (3.1) and (3.47) as

$$L = v[u_{tt} - ku_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu)]. \qquad (3.48)$$

(i) We first consider the Lie point symmetry generator $X_1 = \partial_t$. It can easily be seen from (1.42) that the operator Y_1 is the same as X_1 and that the Lie characteristic function $W = -u_t$. Thus, by using (1.44), the components T^i, $i = 1, 2$, of the conserved vector $T = (T^1, T^2)$ are given by

$$T^1 = v\big(- ku_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu)\big) + u_t v_t, \quad T^2 = -ku_t v_x + kvu_{tx}.$$

Remark. The conserved vector T contains the arbitrary solution v of the adjoint equation (3.47) and hence gives an infinite number of conservation laws. This remark also applies to the two cases given below.

(ii) For the symmetry $X_2 = \partial_x$, we have $W = -u_x$. Thus, by using (1.44), the symmetry generator X_2 gives rise to the following components of the conserved vector:

$$T^1 = v_t u_x - vu_{tx}, \quad T^2 = v\big(u_{tt} + 2\alpha \sinh(nu) + \beta \sinh(2nu)\big) - kv_x u_x.$$

(iii) The symmetry $X_3 = x\,\partial_t + kt\,\partial_x$ has the Lie characteristic function $W = -xu_t - ktu_x$. Thus, invoking (1.44), we obtain the conserved vector T, given by

$$\begin{aligned} T^1 &= xv\big(- ku_{xx} + 2\alpha \sinh(nu) + \beta \sinh(2nu)\big) + xv_t u_t + ktv_t u_x - kvu_x - ktvu_{tx}, \\ T^2 &= ktv\big(u_{tt} + 2\alpha \sinh(nu) + \beta \sinh(2nu)\big) - kxv_x u_t - k^2 tv_x u_x + kvu_t + kxvu_{tx}. \end{aligned}$$

3.2.4 Application of the multiplier method

In this subsection we utilize the multiplier method [50] to obtain conservation laws of the generalized double sinh-Gordon equation (3.1). After some straightforward but lengthy calculations we obtain a single multiplier for (3.1) viz.,

$$\Lambda = \Lambda(t, x, u, u_x) = u_x.$$

Hence the conserved vector $T = (T^1, T^2)$ associated with the above multiplier is given by

$$\begin{aligned} T^1 &= \frac{1}{2}\left(u_t u_x - u u_{tx}\right), \\ T^2 &= \frac{1}{2}\left(u_{tt} u + \frac{2\alpha}{n}\cosh(nu) + \frac{\beta}{n}\cosh(2nu) - k{u_x}^2 - 2\alpha\right). \end{aligned}$$

3.3 Concluding remarks

In this chapter we performed symmetry reductions of the generalized double sinh-Gordon equation (3.1) based on the optimal systems of one-dimensional subalgebras of (3.1). Thereafter, exact solutions with the help of simplest equation method and exponential function method were obtained. These exact solutions obtained here are different from the ones obtained in [13, 16, 44]. Finally, conservation laws for (3.1) were derived by employing four different methods; the direct method, the Noether theorem, the new conservation theorem and multiplier method. The usefulness of conservation laws was discussed in the introduction.

Chapter 4

Exact solutions and conservation laws for a generalized double combined sinh-cosh-Gordon equation

The double combined sinh-cosh-Gordon equation [2,7,45]

$$u_{tt} - ku_{xx} + \alpha \sinh(u) + \alpha \cosh(u) + \beta \sinh(2u) + \beta \cosh(2u) = 0 \tag{4.1}$$

is a well known NLPDE which appears in a wide range of physical applications. It admits geometric interpretation as the differential equation which determines time-like surfaces of constant positive curvature in the same spaces. The travelling wave solutions of this equation were derived by Wazwaz [2] using the tanh method and variable separated ODE method. In [7], (G'/G)-expansion method was used to obtain solutions which were hyperbolic functions and trigonometric functions. Exponential function method was used in [45] to compute travelling wave solutions of (4.1). In this chapter, we study the generalized form of the double combined sinh-cosh-Gordon equation (4.1), given by

$$u_{tt} - ku_{xx} + \alpha \sinh(nu) + \alpha \cosh(nu) + \beta \sinh(2nu) + \beta \cosh(2nu) = 0, \tag{4.2}$$

where k, α and β are non-zero real constants and n is a positive integer.

The aims of this chapter are two fold. Firstly, we use Lie group method together with the simplest equation method to construct exact solutions of (4.2). The second interest is to find conservation laws of (4.2) by using four different approaches namely, the direct method, the Noether theorem, the new conservation theorem and the multiplier method.

This work is new and has been submitted for publication. See [68].

4.1 Symmetry reductions and exact solutions of (4.2)

The symmetry group of the generalized double combined sinh-cosh-Gordon equation (4.2) will be generated by the vector field of the form

$$X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u.$$

The application of the second prolongation $X^{[2]}$ to (4.2) yields the following overdetermined system of linear partial differential equations:

$$\begin{aligned}
&\xi_u = 0,\ \tau_u = 0,\ \eta_{uu} = 0,\ \xi_t - k\tau_x = 0,\\
&\tau_t - \xi_x = 0,\ \tau_{tt} - k\tau_{xx} - 2\eta_{tu} = 0,\ \xi_{tt} - k\xi_{xx} + 2k\eta_{xu} = 0,\\
&\alpha n\,\cosh(nu)\,\eta + \alpha n\,\sinh(nu)\,\eta - k\eta_{xx} - 2\beta\,\tau_t - 2\beta n\,\eta + 2\alpha\,\cosh(nu)\,\tau_t\\
&+2\alpha\,\sinh(nu)\,\tau_t + 4\,\beta\,\cosh^2(nu)\,\tau_t - \alpha\,\cosh(nu)\,\eta_u - \alpha\,\sinh(nu)\,\eta_u\\
&-2\beta\,\cosh^2(nu)\,\eta_u + 4\beta n\,\cosh^2(nu)\,\eta + 4\beta\,\cosh(nu)\sinh(nu)\,\tau_t\\
&-2\beta\,\cosh(nu)\sinh(nu)\,\eta_u + 4\beta n\,\cosh(nu)\sinh(nu)\,\eta_u + \eta_{tt} + \beta\eta_u = 0.
\end{aligned}$$

Solving the above equations we obtain the following three Lie point symmetries:

$$X_1 = \partial_x,\ X_2 = \partial_t,\ X_3 = kt\partial_x + x\partial_t.$$

4.1.1 One-dimensional optimal system of subalgebras

We note that the Lie point symmetries of equation (4.2) are exactly the same as of equation (3.1), namely

$$X_1 = \partial_x,\ X_2 = \partial_t,\ X_3 = kt\partial_x + x\partial_t.$$

Thus, we conclude that the one-dimensional optimal system of subalgebras will be the same as for equation (3.1), namely $\{cX_1 + X_2, X_2, X_3\}$, where c is a non-zero constant.

4.1.2 Symmetry reductions of (4.2)

Here the optimal system of one-dimensional subalgebras $\{cX_1 + X_2, X_2, X_3\}$ will be used to obtain symmetry reductions that transform (4.2) into ordinary differential equations. Thereafter, we will obtain the exact solutions of (4.2).

Case 1. $cX_1 + X_2$

The symmetry generator $cX_1 + X_2$ gives rise to the group-invariant solution

$$u = V(z), \tag{4.3}$$

where $z = x - ct$ is an invariant of the symmetry $cX_1 + X_2$ and V is an arbitrary function of z. The insertion of (4.3) into (4.2) yields the ODE

$$\begin{aligned}(c^2 - k)V''(z) + \alpha\sinh(nV(z)) + \alpha\cosh(nV(z)) + \beta\sinh(2nV(z))\\ +\beta\cosh(2nV(z)) = 0.\end{aligned} \tag{4.4}$$

Using the transformation

$$V(z) = \frac{1}{n}\ln(H(z)) \tag{4.5}$$

on (4.4) we obtain the nonlinear second-order ODE

$$(c^2 - k)H(z)H''(z) - (c^2 - k)H'(z)^2 + \alpha nH(z)^3 + \beta nH(z)^4 = 0. \tag{4.6}$$

The integration of the above equation and reverting back to the original variables, yields the solution of (4.2) in the form

$$x - ct + c_2 = \pm \int \left[\frac{2}{k - c^2} \left(\alpha n \exp(3nu) + \frac{1}{2} \beta n \exp(4nu) \right) + c_1 \exp(2nu) \right]^{-\frac{1}{2}} \times n \exp(nu) du,$$

where c_1 and c_2 are constants of integration.

Case 2. X_2

The symmetry operator X_2 results in the group-invariant solution of the form

$$u = V(z), \tag{4.7}$$

where $z = x$ is an invariant of X_2 and V is an arbitrary function satisfying the ODE

$$kV''(z) - \alpha \sinh(nV(z)) - \alpha \cosh(nV(z)) - \beta \sinh(2nV(z)) - \beta \cosh(2nV(z)) = 0. \tag{4.8}$$

The transformation (4.5) on equation (4.8) yields

$$kH(z)H''(z) - kH'(z)^2 - \alpha n H(z)^3 - \beta n H(z)^4 = 0. \tag{4.9}$$

Solving the above equation and reverting back to the original variables, yields

$$x + c_2 = \pm \int \left[\frac{2}{k} \left(\alpha n \exp(3nu) + \frac{1}{2} \beta n \exp(4nu) \right) + c_1 \exp(2nu) \right]^{-\frac{1}{2}} n \exp(nu) du,$$

where c_1 and c_2 are constants of integration.

Case 3. X_3

The symmetry X_3 gives rise to the group-invariant solution

$$u = V(z), \tag{4.10}$$

where $z = x^2 - kt^2$ is an invariant of X_3 and the arbitrary function V satisfies the ODE

$$4kzV''(z) + 4kV'(z) - \alpha \sinh(nV(z)) - \alpha \cosh(nV(z)) - \beta \sinh(2nV(z)) - \beta \cosh(2nV(z)) = 0.$$

4.1.3 Exact solutions using simplest equation method

In this subsection we invoke the simplest equation method to solve the highly nonlinear ODE (4.6). This will then give us the exact solutions for the generalized double combined sinh-cosh-Gordon equation (4.2). The simplest equations that we will use here are the Bernoulli and Riccati equations.

Solutions of (4.2) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 1$, so the solutions of (4.6) take the form

$$H(z) = A_0 + A_1 G. \tag{4.11}$$

Inserting (4.11) into (4.6) and using the Bernoulli equation [69] and thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of five equations in terms of A_0, A_1, namely

$$\begin{aligned}
\beta n A_0^4 + \alpha n A_0^3 &= 0, \\
\beta n A_1^4 - b^2 k A_1^2 + b^2 c^2 A_1^2 &= 0, \\
-ka^2 A_0 A_1 + 4\beta n A_0^3 A_1 + 3\alpha n A_0^2 A_1 + a^2 c^2 A_0 A_1 &= 0, \\
-3abk A_0 A_1 + 3abc^2 A_0 A_1 + 6\beta n A_0^2 A_1^2 + 3\alpha n A_0 A_1^2 &= 0, \\
\alpha n A_1^3 + abc^2 A_1^2 + 2b^2 c^2 A_0 A_1 - 2b^2 k A_0 A_1 - abk A_1^2 + 4\beta n A_0 A_1^3 &= 0.
\end{aligned}$$

With the aid of Maple, solving the above system of algebraic equations, one possible solution is

$$A_0 = \frac{a^2\left(c^2 - k\right)}{\alpha n}, \quad A_1 = \frac{ab\left(c^2 - k\right)}{\alpha n}, \quad \beta = -\frac{\alpha^2 n}{a^2\left(c^2 - k\right)}.$$

Thus, reverting back to the original variables, a solution of (4.2) is given by

$$u(t,x) = \frac{1}{n}\ln\left[A_0 + A_1 a\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\right], \tag{4.12}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

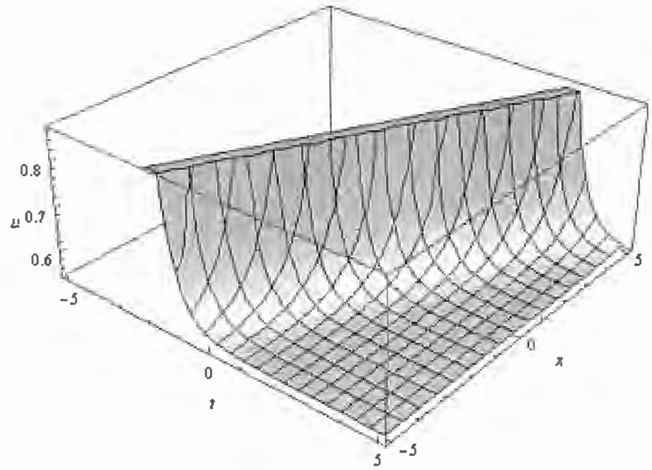

Figure 4.1: Profile of the solution (4.12)

Solutions of (4.2) using the Riccati equation as the simplest equation

The balancing procedure yields $M = 1$, so the solutions of (4.6) take the form

$$H(z) = A_0 + A_1 G. \tag{4.13}$$

Inserting (4.13) into (4.6) and making use of the Riccati equation [69], we obtain algebraic system of equations in terms of A_0, A_1 by equating the coefficients powers of G^i to zero. The resulting algebraic equations are

$$\begin{aligned}
\beta n A_0^4 + \alpha n A_0^3 + k\nu^2 A_1^2 - c^2\nu^2 A_1^2 + bc^2\nu A_0 A_1 - bk\nu A_0 A_1 &= 0,\\
-a^2 k A_1^2 + a^2 c^2 A_1^2 + \beta n A_1^4 &= 0,\\
3\alpha n A_0 A_1^2 - 3abk A_0 A_1 + 3abc^2 A_0 A_1 + 6\beta n A_0^2 A_1^2 &= 0,\\
-abk A_1^2 - 2a^2 k A_0 A_1 + abc^2 A_1^2 + 4\beta n A_0 A_1^3 + \alpha n A_1^3 + 2a^2 c^2 A_0 A_1 &= 0,\\
2ac^2\nu A_0 A_1 + 4\beta n A_0^3 A_1 + 3\alpha n A_0^2 A_1 - b^2 k A_0 A_1 + bk\nu A_1^2 + b^2 c^2 A_0 A_1 &\\
-2ak\nu A_0 A_1 - bc^2\nu A_1^2 &= 0.
\end{aligned}$$

Solving the above equations, we get

$$A_0 = \frac{A_1(b + \sqrt{b^2 - 4a\nu})}{2a}, \quad \alpha = \frac{(bA_0 - \nu A_1)(c^2 b A_0 + 2k\nu A_1 - 2c^2\nu A_1 - kbA_0)}{nA_0^3},$$

$$\beta = -\frac{(c^2 - k)(bA_0 - \nu A_1)^2}{nA_0^4}$$

and consequently, the solutions of (4.2) are given by

$$u(t,x) = \frac{1}{n}\ln\left[A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\right] \tag{4.14}$$

and

$$u(t,x) = \frac{1}{n}\ln\left[A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}\right],$$

where $z = x - ct$ and C is an arbitrary constant of integration.

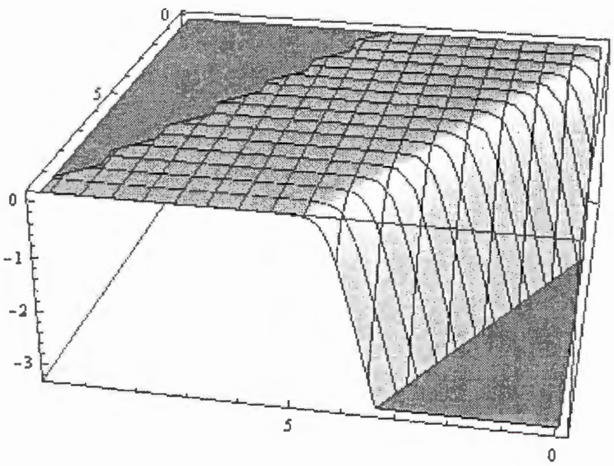

Figure 4.2: Profile of the solution (4.14)

4.2 Construction of conservation laws of (4.2)

In this section conservation laws will be constructed for the generalized double combined sinh-cosh-Gordon equation (4.2) by four different methods, namely, the direct method, the Noether theorem the new conservation theorem and the multiplier method.

We recall (4.2) admits the following three Lie point symmetry generators:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = kt\partial_x + x\partial_t.$$

4.2.1 Application of the direct method

There is a fundamental relationship between the point symmetries admitted by a given equation and the conservation laws of that given equation. Following [39], we see that the conservation law

$$D_tT^1 + D_xT^2 = 0, \tag{4.15}$$

which must be evaluated on the partial differential equation, can be considered together with the following requirements

$$X^{[n]}(T^1) + T^1D_x(\xi) - T^2D_x(\tau) = 0, \tag{4.16}$$

$$X^{[n]}(T^2) + T^2D_t(\tau) - T^1D_t(\xi) = 0, \tag{4.17}$$

in which $X^{[n]}$ is the nth prolongation of a point symmetry of the original equation. The order of the extension equals to the order of the highest derivative in T^1 and T^2. Consequently, for given X, (4.15)-(4.17) can be solved to obtain the conserved vectors or tuple $T = (T^1, T^2)$.

The condition (4.15) on the equation (4.2) gives

$$\Big(ku_{xx} - \alpha\sinh(nu) - \alpha\cosh(nu) - \beta\sinh(2nu) - \beta\cosh(2nu)\Big)\frac{\partial T^1}{\partial u_t} +$$
$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + u_{tx}\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + u_{tx}\frac{\partial T^2}{\partial u_t} + u_{xx}\frac{\partial T^2}{\partial u_x} = 0.$$

Since T^1 and T^2 are independent of the second derivatives of u, it implies that the coefficients of u_{tt}, u_{tx} and u_{xx} must be zero. Hence,

$$\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial u_t} = 0, \tag{4.18}$$

$$k\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial u_x} = 0, \tag{4.19}$$

$$\frac{\partial T^1}{\partial t} - \Big(\alpha\sinh(nu) + \alpha\cosh(nu) + \beta\sinh(2nu) + \beta\cosh(2nu)\Big)\frac{\partial T^1}{\partial u_t}$$
$$+u_t\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} = 0. \tag{4.20}$$

We now construct the conservation laws for the generalized double combined sinh-cosh-Gordon equation (4.2) using the three admitted Lie point symmetries.

We start with the translation symmetry

$$X_1 = \partial_t, \tag{4.21}$$

which is already in its extended form. The symmetry conditions (4.16)-(4.17) yield

$$\frac{\partial T^1}{\partial t} = 0, \tag{4.22}$$

$$\frac{\partial T^2}{\partial t} = 0, \tag{4.23}$$

respectively. Therefore from (4.18)-(4.20) and (4.22)-(4.23) the components of the conserved vector of the generalized double combined sinh-cosh-Gordon equation (4.2) associated with the symmetry X_1 are given by

$$\begin{aligned} T^1 &= \frac{c_4}{nk}\Big[2\alpha\cosh(nu) + 2\alpha\sinh(nu) + \beta\cosh(2nu) + \beta\sinh(2nu)\Big] \\ &\quad + \frac{c_4 u_t^2}{k} + c_4 u_x^2 + c_5 u_x + j(x) + c_6, \\ T^2 &= -2c_4 u_t u_x - c_5 u_t + c_7, \end{aligned}$$

where c_4, c_5, c_6 and c_7 are arbitrary constants and $j(x)$ is a function of x.

Continuing in the same manner using X_2 and X_3 we obtain the components of the conserved vector for equation (4.2) as

$$\begin{aligned} T^1 &= \frac{c_4}{nk}\Big[2\alpha\cosh(nu) + 2\alpha\sinh(nu) + \beta\cosh(2nu) + \beta\sinh(2nu)\Big] \\ &\quad + \frac{c_4 u_t^2}{k} + c_4 u_x^2 + c_5 u_x + c_8 \\ T^2 &= -2c_4 u_t u_x - c_5 u_t + p(t), \end{aligned}$$

and

$$\begin{aligned} T^1 &= c_5 u_x + c_9 x, \\ T^2 &= -c_5 u_t + c_9 kt, \end{aligned}$$

respectively, where c_4, c_5, c_8 and c_9 are constants and $p(t)$ is a function of t. We note that the symmetry X_3 gives a trivial conserved vector.

4.2.2 Application of the Noether theorem

We now apply the Noether theorem to construct conservation laws of the generalized double combined sinh-cosh-Gordon equation (4.2). It can be easily verified that equation (4.2) has a first-order Lagrangian given by

$$L=\frac{1}{2}u_t^2-\frac{k}{2}u_x^2-\frac{1}{2n}\Big(2\alpha\cosh(nu)+2\alpha\sinh(nu)+\beta\cosh(2nu)+\beta\sinh(2nu)\Big). \quad (4.24)$$

The insertion of (4.24) into the Noether operator determining equation (1.37) gives

$$\begin{aligned}&-\alpha\eta\sinh(nu)-\alpha\eta\cosh(nu)-\beta\eta\sinh(2nu)-\beta\eta\cosh(2nu)+u_t\Big\{\eta_t+u_t\eta_u-\\&u_t(\tau_t+u_t\tau_u)-u_x(\xi_t+u_t\xi_u)\Big\}-ku_x\Big\{\eta_x+u_x\eta_u-u_t(\tau_x+u_x\tau_u)-u_x(\xi_x+u_x\xi_u)\Big\}\\&+\Big\{(1/2)u_t^2-(k/2)u_x^2-\frac{\alpha\cosh(nu)}{n}-\frac{\alpha\sinh(nu)}{n}-\frac{\beta\cosh(2nu)}{2n}-\frac{\beta\sinh(2nu)}{2n}\Big\}\\&\Big\{\tau_t+u_t\tau_u+\xi_x+u_x\xi_u\Big\}=B_t^1+u_tB_u^1+B_x^2+u_xB_u^2.\end{aligned}$$

The separation of the above equation on the derivatives of u, yields

$$\begin{aligned}\tau_u=0,\ \xi_u=0,\ k\tau_x-\xi_t=0,\ \xi_x-\tau_t+2\eta_u=0,\\ \xi_x-2\eta_u-\tau_t=0,\ \eta_t-B_u^1,0,\ k\eta_x+B_u^2=0,\\ \alpha\eta\sinh(nu)+\alpha\eta\cosh(nu)+\beta\eta\sinh(2nu)+\beta\eta\cosh(2nu)+\{(\alpha/n)\cosh(nu)\}\tau_t\\ +\{(\alpha/n)\sinh(nu)\}\tau_t+\{(\beta/2n)\cosh(2nu)\}\tau_t+\{(\beta/2n)\sinh(2nu)\}\tau_t+\{(\alpha/n)\cosh(nu)\}\xi_x\\ +\{(\alpha/n)\sinh(nu)\}\xi_x+\{(\beta/2n)\cosh(2nu)\}\xi_x+\{(\beta/2n)\sinh(2nu)\}\xi_x+B_t^1+B_x^2=0.\end{aligned}$$

Solving the above equations yield

$$\tau=d_1,\ \xi=-d_2,\ \eta=0,$$
$$B_t^1(t,x)+B_x^2(t,x)=0,$$

where d_1 and d_2 are arbitrary constants and $B^1(t,x)$ and $B^2(t,x)$ are arbitrary functions of t and x. We can take $B^1(t,x)=B^2(t,x)=0$ since they contribute to the trivial part of the conserved vectors. Thus, we get two Noether point symmetries, namely

$$X_1=\partial_t,\quad X_2=\partial_x.$$

The use of the theorem due to Noether, with $X_1 = \partial_t$, gives the conserved vector

$$T^1 = -\frac{1}{2}u_t^2 - \frac{k}{2}u_x^2 - \frac{1}{2n}\Big[2\alpha\cosh(nu) + 2\alpha\sinh(nu) + \beta\cosh(2nu) + \beta\sinh(2nu)\Big],$$
$$T^2 = ku_tu_x.$$

Using $X_2 = \partial_x$ and employing the Noether theorem, we obtain

$$T^1 = -u_tu_x,$$
$$T^2 = \frac{1}{2}u_t^2 + \frac{k}{2}u_x^2 - \frac{1}{2n}\Big[2\alpha\cosh(nu) + 2\alpha\sinh(nu) + \beta\cosh(2nu) + \beta\sinh(2nu)\Big].$$

4.2.3 Application of the new conservation theorem

In this subsection we use the new conservation theorem [49] and construct conservation laws for (4.2). The adjoint equation of (4.2), by invoking (1.40), is

$$\begin{aligned} E^*(t,x,u,v,\ldots,u_{xx},v_{xx}) &= \frac{\delta}{\delta u}\Big[v\{u_{tt} - ku_{xx} + \alpha\sinh(nu) + \alpha\cosh(nu) \\ &\quad +\beta\sinh(2nu) + \beta\cosh(2nu)\}\Big] = 0, \end{aligned} \qquad (4.25)$$

where $v = v(t,x)$ is a new dependent variable. Thus from (4.25) we have

$$v_{tt} - kv_{xx} + nv\{\alpha\sinh(nu) + \alpha\cosh(nu) + 2\beta\sinh(2nu) + 2\beta\cosh(2nu)\} = 0. \qquad (4.26)$$

It is clear from the adjoint equation (4.26) that equation (4.2) is not self-adjoint. By recalling (1.45), we obtain the following Lagrangian for the system of equations (4.2) and (4.26):

$$L = v\{u_{tt} - ku_{xx} + \alpha\sinh(nu) + \alpha\cosh(nu) + \beta\sinh(2nu) + \beta\cosh(2nu)\}. \qquad (4.27)$$

(i) We first consider the Lie point symmetry generator $X_1 = \partial_t$. It can easily be shown from (1.42) that the operator Y_1 is the same as X_1 and hence the Lie characteristic function $W = -u_t$. Therefore by using (1.44), the components T^i, $i = 1, 2$, of the conserved vector $T = (T^1, T^2)$ are

$$T^1 = v\Big(-ku_{xx} + \alpha\sinh(nu) + \alpha\cosh(nu) + \beta\sinh(2nu) + \beta\cosh(2nu)\Big) + u_tv_t,$$

$T^2 = kvu_{tx} - ku_t v_x.$

Remark. The conserved vector T contains the arbitrary solution v of the adjoint equation (4.26) and hence gives an infinite number of conservation laws.

This remark also applies to the following two cases.

(ii) The generator $X_2 = \partial_x$, gives $W = -u_x$. Thus, by using (1.44), the generator X_2 we have the following components of the conserved vector:

$$\begin{aligned} T^1 &= v_t u_x - v u_{tx}, \\ T^2 &= v\Big(u_{tt} + \alpha \sinh(nu) + \alpha \cosh(nu) + \beta \sinh(2nu) + \beta \cosh(2nu)\Big) - kv_x u_x. \end{aligned}$$

(iii) The symmetry generator $X_3 = x\partial_t + kt\partial_x$ has the Lie characteristic function $W = -xu_t - ktu_x$. Thus, invoking (1.44), we obtain the conserved vector T as

$$\begin{aligned} T^1 &= xv\Big(- ku_{xx} + \alpha \sinh(nu) + \alpha \cosh(nu) + \beta \sinh(2nu) + \beta \cosh(2nu)\Big) \\ &\quad + xv_t u_t + ktv_t u_x - kvu_x - ktvu_{tx}, \\ T^2 &= ktv\Big(u_{tt} + \alpha \sinh(nu) + \alpha \cosh(nu) + \beta \sinh(2nu) + \beta \cosh(2nu)\Big) \\ &\quad - kxv_x u_t - k^2 tv_x u_x + kvu_t + kxvu_{tx}. \end{aligned}$$

4.2.4 Application of the multiplier method

Lastly, in this subsection we utilize the multiplier method [50] to obtain conservation laws of (4.2). After some straightforward but lengthy calculations we obtain the first-order multiplier for (4.2), viz.,

$$\Lambda = \Lambda(t, x, u, u_t, u_x) = C_1(xu_t + ktu_x) + C_2 u_x + C_3 u_t,$$

where $C_i, i = 1, 2, 3$ are arbitrary constants. The above multiplier yields the following three local conserved vectors of (4.2). Hence the conserved vectors associated with the above multiplier are given by

$$T_1^1 = \frac{1}{2}\Big(- ku_{xx}xu - ku_{tx}tu - ku_x u + u_t^2 x + ku_t u_x t + \frac{2\alpha x}{n}\sinh(nu) + \frac{2\alpha x}{n}\cosh(nu)$$

$$
\begin{aligned}
& +\frac{2\beta x}{n}\cosh(nu)\sinh(nu)+\frac{2\beta x}{n}\cosh^2(nu)-\frac{2\alpha x}{n}-\frac{2\beta x}{n}\Bigg), \\
T_1^2 &= \frac{k}{2}\Bigg(u_{tt}tu-u_tu_xx+u_tu-ku_x^2t+u_{tx}xu+\frac{2\alpha t}{n}\sinh(nu)+\frac{2\alpha t}{n}\cosh(nu) \\
& +\frac{2\beta t}{n}\cosh(nu)\sinh(nu)+\frac{2\beta t}{n}\cosh^2(nu)-\frac{2\alpha t}{n}-\frac{2\beta t}{n}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
T_2^1 &= \frac{1}{2}(u_tu_x-uu_{tx}), \\
T_2^2 &= \frac{1}{2}\Bigg(u_{tt}u-ku_x^2+\frac{2\alpha}{n}\sinh(nu)+\frac{2\alpha}{n}\cosh(nu)+\frac{2\beta}{n}\cosh(nu)\sinh(nu) \\
& +\frac{2\beta}{n}\cosh^2(nu)-\frac{2\alpha}{n}-\frac{2\beta}{n}\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
T_3^1 &= \frac{1}{2}\Bigg(-ku_{xx}u+u_t^2+\frac{2\alpha}{n}\sinh(nu)+\frac{2\alpha}{n}\cosh(nu)+\frac{2\beta}{n}\cosh(nu)\sinh(nu) \\
& +\frac{2\beta}{n}\cosh^2(nu)-\frac{2\alpha}{n}-\frac{2\beta}{n}\Bigg), \\
T_3^2 &= \frac{k}{2}(uu_{tx}-u_tu_x).
\end{aligned}
$$

4.3 Concluding remarks

The generalized double combined sinh-cosh-Gordon equation (4.2) was investigated by using the Lie symmetry analysis. Symmetry reductions based on the optimal systems of one-dimensional subalgebras of (4.2) were obtained. Thereafter exact solutions with the help of simplest equation method were constructed. The exact solutions obtained here are more general than the ones given in [2, 7, 45]. Futhermore, several conserved quantities for (4.2) were derived by employing four different techniques; the direct method, the Noether theorem, the new conservation theorem and the multiplier method.

Chapter 5

Exact solutions and conservation laws for the (2+1)-dimensional nonlinear sinh-Gordon equation

The sinh-Gordon equation

$$u_{tt} - u_{xx} + \sinh u = 0$$

is one of the equations which appears in solitary waves theory [46]. This equation gained importance because of its collisional behaviours of solitons. It first appeared in the propagation of fluxons in the Josephson junction [47] between two superconductors. In (2+1) dimensions the sinh-Gordon equation is given by

$$u_{tt} - u_{xx} - u_{yy} + \sinh u = 0. \tag{5.1}$$

Equation (5.1) has applications in solid state physics, integrable field theory, kink dynamics, fluid dynamics, and many other scientific fields. This equation was studied using numerical and analytical approaches and more recently one soliton solution and two soliton solutions were formally derived using the simplified Hirota's method in [46].

In this chapter we study equation (5.1). The objectives of this chapter are in two directions. Firstly, we employ the Lie group analysis along with the simplest equation method and (G'/G)-expansion method to construct exact solutions of (5.1). Secondly, conservation laws for the underlying equation are also derived by employing the direct method, Noether theorem and the new conservation theorem.

This work is new and has been submitted for publication. See [70].

5.1 Symmetry reductions and exact solutions of (5.1)

We first compute the Lie point symmetries of (5.1). Let us assume that the vector field of the form

$$X = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \psi(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u$$

will generate the symmetry group of (5.1). Applying the second prolongation $X^{[2]}$ to (5.1) we obtain an overdetermined system of thirteen linear partial differential equations, namely

$$\begin{aligned}
&\xi_u = 0,\ \tau_u = 0,\ \psi_u = 0,\ \eta_{uu} = 0,\ \xi_t - \tau_x = 0,\ \tau_y - \psi_t = 0,\\
&\xi_y + \psi_x = 0,\ \xi_x - \psi_y = 0,\ \tau_t - \psi_y = 0,\ \tau_{tt} - \tau_{xx} - \tau_{yy} - 2\eta_{tu} = 0,\\
&\xi_{tt} - \xi_{xx} - \xi_{yy} + 2\eta_{xu} = 0,\ \psi_{tt} - \psi_{xx} - \psi_{yy} + 2\eta_{yu} = 0,\\
&\eta\cosh(u) + 2\,\psi_y\sinh(u) - \eta_u\sinh(u) + \eta_{tt} - \eta_{xx} - \eta_{yy} = 0.
\end{aligned}$$

After some tedious calculations one can obtain the values of τ, ξ, ψ and η. Thus equation (5.1) has the following six Lie point symmetries:

$$\begin{aligned}
&X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = x\partial_t + t\partial_x\\
&X_5 = y\partial_t + t\partial_y,\ X_6 = -y\partial_x + x\partial_y.
\end{aligned}$$

We now use the three translation symmetries X_1, X_2 and X_3 to transform (5.1) to a partial differential equation of two independent variables. The linear combination of these three symmetry, namely $X = X_1 + X_2 + X_3$ gives the following three invariants:

$$p = t - y,\ q = x - y,\ v = u. \tag{5.2}$$

Taking p and q as new independent variables and v as the new dependent variable, the equation (5.1) transforms to

$$2v_{qq} + 2v_{pq} - \sinh(v) = 0. \tag{5.3}$$

Now we further reduce (5.3) using its symmetries. The equation (5.3) has the following three symmetries:

$$V_1 = \partial_p, \quad V_2 = \partial_q, \quad V_3 = (p - 2q)\partial_p - \partial_q.$$

The symmetry $cV_1 + V_2$, where $c \neq 0$ is a constant, yields the invariants $z = p - c\,q$ and $\phi = v$, which gives the group-invariant solution $\phi = \phi(z)$ where ϕ satisfies the second-order nonlinear ODE

$$(2c - 2c^2)\phi''(z) + \sinh(\phi) = 0. \tag{5.4}$$

Using the transformation

$$\phi(z) = \ln(w(z)) \tag{5.5}$$

on (5.4) we get

$$(4c - 4c^2)w''(z)w(z) + (4c^2 - 4c)w'(z)^2 + w(z)^3 - w(z) = 0. \tag{5.6}$$

5.1.1 Exact solutions using simplest equation method

In this subsection we invoke the simplest equation method [65] to solve the highly nonlinear ODE (5.6). Consequently this will then give us the exact solutions for the (2+1)-dimensional nonlinear sinh-Gordon equation (5.1). The simplest equations that we use are the Bernoulli and Riccati equations [69].

We assume that the solutions of the nonlinear ODE (5.6) are of the form

$$w(z) = \sum_{i=0}^{M} A_i (G(z))^i, \tag{5.7}$$

where $G(z)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by the balancing procedure and $A_0, \cdots, A_M$ are parameters to be determined.

Solutions of (5.1) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of (5.6) are of the form

$$w(z) = A_0 + A_1 G + A_2 G^2. \tag{5.8}$$

This value of $w(z)$ is now inserted in (5.6). Then using the Bernoulli equation [69] and thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of seven equations in terms of A_0, A_1, A_2, namely

$$\begin{aligned}
A_0 - A_0^3 &= 0,\\
4kb^2A_2^2 + A_2^3 &= 0,\\
3A_0^2A_1 + 2ka^2A_0A_1 - A_1 &= 0,\\
8kb^2A_1A_2 + 4kabA_2^2 + 3A_1A_2^2 &= 0,\\
6kabA_0A_1 - 3A_0^2A_2 + 8ka^2A_0A_2 - A_2 + 3A_0A_1^2 &= 0,\\
3A_1^2A_2 + 10kabA_1A_2 + kb^2A_1^2 + 12kb^2A_0A_2 + 3A_0A_2^2 &= 0,\\
4kA_0A_2b^2 + 20kabA_0A_2 + 2ka^2A_1A_2 + 6A_0A_1A_2 + 2kabA_1^2 + A_1^3 &= 0.
\end{aligned}$$

With the aid of Maple, we solve the above system of algebraic equations and obtain

$$a = k,\ A_0 = 1,\ A_1 = 8bkc^2 - 8bkc,\ A_2 = 8b^2c^2 - 8b^2c,$$

where k is any root of $(2c^2 - 2c)k^2 - 1 = 0$. Thus, from (5.8) and (5.5), and then reverting back to the original variables, a solution of (7.1) is [69]

$$\begin{aligned}
u(t,x,y) = \ln\Bigg[A_0 + aA_1\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\\
+ a^2A_2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2\Bigg],
\end{aligned} \tag{5.9}$$

where $z = t - cx + (c-1)y$ and C is an arbitrary constant of integration.

Solutions of (5.1) using the Riccati equation as the simplest equation

The balancing procedure gives $M = 2$, and so (5.7) becomes

$$w(z) = A_0 + A_1 G + A_2 G^2. \tag{5.10}$$

The insertion of this value of $w(z)$ into (5.6) and making use of the Riccati equation [69], yields the following algebraic system of equations in terms of A_0, A_1, A_2:

$$A_2^3 + 4ka^2A_2^2 = 0,$$
$$3A_1A_2^2 + 8ka^2A_1A_2 + 2kabA_2^2 = 0,$$
$$10kabA_1A_2 + 2ka^2A_1^2 + 12ka^2A_0A_2 + 3A_1^2A_2 + 3A_0A_2^2 = 0,$$
$$2kabA_1^2 + A_1^3 + 6A_0A_1A_2 + 20kabA_0A_2 + 4ka\nu A_0A_2 + 2kb^2A_1A_2 - 4kb\nu A_2^2 + 4ka^2A_0A_1 = 0,$$
$$16ka\nu A_0A_2 - 4k\nu^2A_2^2 + 3A_0A_1^2 - A_2 + 8kb^2A_0A_2 + 6kabA_0A_1 + 3A_0^2A_2 - 2kb\nu A_1A_2 = 0,$$
$$12kb\nu A_0A_2 - 4k\nu^2A_1A_2 - 2kb\nu A_1^2 + 2kb^2A_0A_1 + 3A_0^2A_1 + ka\nu A_0A_1 - A_1 = 0,$$
$$2kb\nu A_0A_1 + A_0^3 + 4k\nu^2A_0A_2 - 2k\nu^2A_1^2 - A_0 = 0.$$

The solution of the above system using Maple is

$$b = \rho,\ A_0 = 8a\nu c^2 - 8a\nu c - 1,\ A_1 = 8a\rho c^2 - 8a\rho c,\ A_2 = 8a^2c^2 - 8a^2c,$$

where ρ is any root of $(2c^2-2c)\rho^2 - 8a\nu c^2 + 8a\nu c + 1 = 0$. Consequently, the solutions of (5.1) are [69]

$$u(t,x,y) = \ln\left[A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z+C)\right)\right\} + A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta(z+C)\right)\right\}^2\right] \tag{5.11}$$

and

$$u(t,x,y) = \ln\left[A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} + A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2\right], \tag{5.12}$$

where $z = t - cx + (c-1)y$ and C is an arbitrary constant of integration.

5.1.2 Solutions of (5.1) using (G'/G)-expansion method

In this subsection we use the $(G'/G)-$expansion method [8] and obtain some exact solutions of the ODE (5.6). This will result in the exact solutions of the equation

(5.1).

Let us consider the solutions of (5.6) in the form

$$w(z)=\sum_{i=0}^{M}\alpha_i\left(\frac{G'(z)}{G(z)}\right)^i, \tag{5.13}$$

where $G(z)$ satisfies the linear second-order ODE with constant coefficients, viz.,

$$G''+\lambda G'+\mu G=0, \tag{5.14}$$

where λ and μ are constants. The positive integer M in (5.13) is found by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (5.6). The coefficients $\alpha_0,\cdots,\alpha_M$ are parameters to be determined.

The application of the balancing procedure to the ODE yields $M=2$, so the solutions of (5.6) are of the form

$$w(z)=\alpha_0+\alpha_1\left(\frac{G'(z)}{G(z)}\right)+\alpha_2\left(\frac{G'(z)}{G(z)}\right)^2. \tag{5.15}$$

Inserting the value of $w(z)$ from (5.15) into (5.6) and making use of (5.14) leads to the following overdetermined system of algebraic equations:

$$\alpha_0^3-\alpha_0-4\alpha_1\alpha_0\lambda\mu c^2-8\alpha_2\alpha_0\mu^2c^2+4\alpha_1^2\mu^2c^2+4\alpha_1\alpha_0\lambda\mu p+8\alpha_2\alpha_0\mu^2c-4\alpha_1^2\mu^2c=0,$$

$$3\alpha_0^2\alpha_1-\alpha_1-4\alpha_0\alpha_1\lambda^2c^2+4\alpha_1^2\lambda\mu c^2-24\alpha_0\alpha_2\lambda\mu c^2+8\alpha_1\alpha_2\mu^2c^2-8\alpha_0\alpha_1\mu c^2$$
$$+4\alpha_0\alpha_1\lambda^2c\mu-4\alpha_1^2\lambda+24\alpha_0\alpha_2\lambda\mu c-8\alpha_1\alpha_2\mu^2c+8\alpha_0\alpha_1\mu c=0,$$

$$3\alpha_0\alpha_1^2+3\alpha_0^2\alpha_2-\alpha_2-16\alpha_0\alpha_2\lambda^2c^2+4\alpha_1\alpha_2\lambda\mu c^2-12\alpha_0\alpha_1\lambda c^2+8\alpha_2^2\mu^2c^2-32\alpha_0\alpha_2\mu c^2$$
$$+16\alpha_0\alpha_2\lambda^2c-4\alpha_1\alpha_2\lambda\mu c+12\alpha_0\alpha_1\lambda c-8\alpha_2^2\mu^2c+32\alpha_0\alpha_2\mu c=0,$$

$$\alpha_1^3+6\alpha_0\alpha_2\alpha_1-4\alpha_2\alpha_1\lambda^2c^2+8\alpha_2^2\lambda\mu c^2-4\alpha_1^2\lambda c^2-40\alpha_0\alpha_2\lambda c^2-8\alpha_2\alpha_1\mu c^2-8\alpha_0\alpha_1c^2$$
$$+4\alpha_2\alpha_1\lambda^2c-8\alpha_2^2\lambda\mu c+4\alpha_1^2\lambda c+40\alpha_0\alpha_2\lambda c+8\alpha_2\alpha_1\mu c+8\alpha_0\alpha_1c=0,$$

$$3\alpha_0\alpha_2^2+3\alpha_1^2\alpha_2-20\alpha_1\alpha_2\lambda c^2-4\alpha_1^2c^2-24\alpha_0\alpha_2c^2+20\alpha_1\alpha_2\lambda c+4\alpha_1^2c+24\alpha_0\alpha_2c=0,$$

$$3\alpha_1\alpha_2^2-8\alpha_2^2\lambda c^2-16\alpha_1\alpha_2c^2+8\alpha_2^2\lambda c+16\alpha_1\alpha_2c=0,$$

$$\alpha_2^3-8\alpha_2^2c^2+8\alpha_2^2c=0.$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$\alpha_0 = 2\left(\lambda^2 c^2 - \lambda^2 c\right)$$
$$\alpha_1 = 8\left(\lambda c^2 - \lambda c\right)$$
$$\alpha_2 = 8\left(c^2 - c\right)$$
$$c = \frac{\lambda^2 - 4\mu + \sqrt{\lambda^4 - 8\lambda^2\mu + 2\lambda^2 + 16\mu^2 - 8\mu}}{2\left(\lambda^2 - 4\mu\right)}.$$

Now using the general solution of (5.14) in (5.15), we have the following two types of travelling wave solutions of equation (5.1):

When $\lambda^2 - 4\mu > 0$, we obtain

$$u_1(t,x,y) = \ln\left[\alpha_0 + \alpha_1\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\right)\right. \tag{5.16}$$
$$\left. + \alpha_2\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\right)^2\right] \tag{5.17}$$

where $z = t - cx + (c-1)y$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain

$$u_2(t,x,y) = \ln\left[\alpha_0 + \alpha_1\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right)\right.$$
$$\left. + \alpha_2\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right)^2\right],$$

where $z = t - cx + (c-1)y$, $\delta_2 = \frac{1}{2}\sqrt{4\mu - \lambda^2}$, C_1 and C_2 are arbitrary constants.

5.2 Conservation laws of (5.1)

We derive conservation laws for the (2+1)-dimensional nonlinear sinh-Gordon equation

$$u_{tt} - u_{xx} - u_{yy} + \sinh u = 0,$$

by using three different methods, namely, the direct method, the Noether theorem and the new conservation theorem.

5.2.1 Application of the direct method

Following [39], we see that the conservation law

$$D_tT^1 + D_xT^2 + D_xT^3 = 0, \tag{5.18}$$

which must be evaluated on the partial differential equation, can be considered together with the following requirements

$$X^{[n]}(T^1) + T^1(D_x\xi + D_y\psi) - (T^2D_x\tau + T^3D_y\tau) = 0, \tag{5.19}$$

$$X^{[n]}(T^2) + T^2(D_t\tau + D_y\psi) - (T^1D_t\xi + T^3D_y\xi) = 0, \tag{5.20}$$

$$X^{[n]}(T^3) + T^3(D_t\tau + D_x\xi) - (T^1D_t\psi + T^2D_t\psi) = 0, \tag{5.21}$$

in which $X^{[n]}$ is the nth prolongation of a point symmetry of the original equation. The order of the extension equals to the order of the highest derivative in T^1, T^2 and T^3. Consequently, for given X, (5.18)-(5.21) can be solved to obtain the conserved vectors or tuple $T = (T^1, T^2, T^3)$.

The condition (5.18) on the equation (5.1) gives

$$\frac{\partial T^1}{\partial t} + u_t\frac{\partial T^1}{\partial u} + u_{tx}\frac{\partial T^1}{\partial u_x} + u_{ty}\frac{\partial T^1}{\partial u_y} + \left(u_{xx} + u_{yy} - \sinh u\right)\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} +$$
$$u_{tx}\frac{\partial T^2}{\partial u_t} + u_{xx}\frac{\partial T^2}{\partial u_x} + u_{xy}\frac{\partial T^2}{\partial u_y} + \frac{\partial T^3}{\partial y} + u_y\frac{\partial T^3}{\partial u} + u_{xy}\frac{\partial T^3}{\partial u_x} + u_{yy}\frac{\partial T^3}{\partial u_y} + u_{ty}\frac{\partial T^3}{\partial u_t} = 0.$$

Since T^1, T^2 and T^3 are independent of the second derivatives of u, it implies that the coefficients of $u_{xt}, u_{xx}, u_{ty}, u_{yy}$ and u_{xy} must be zero. Hence,

$$\frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial u_t} = 0, \tag{5.22}$$

$$\frac{\partial T^1}{\partial u_t} + \frac{\partial T^2}{\partial u_x} = 0, \tag{5.23}$$

$$\frac{\partial T^1}{\partial u_y} + \frac{\partial T^3}{\partial u_t} = 0, \tag{5.24}$$

$$\frac{\partial T^1}{\partial u_t} + \frac{\partial T^3}{\partial u_y} = 0, \tag{5.25}$$

$$\frac{\partial T^2}{\partial u_y} + \frac{\partial T^3}{\partial u_x} = 0, \tag{5.26}$$

$$\frac{\partial T^1}{\partial t} - \sinh u\frac{\partial T^1}{\partial u_t} + u_t\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + \frac{\partial T^3}{\partial y} + u_y\frac{\partial T^3}{\partial u} = 0. \tag{5.27}$$

We now construct the conservation laws for the (2+1)-dimensional nonlinear sinh-Gordon equation (5.1) using the six admitted Lie point symmetries.

We start with the time translation symmetry

$$X_1 = \partial_t, \tag{5.28}$$

which is already in its extended form. The symmetry conditions (5.19)-(5.21) yield

$$\frac{\partial T^1}{\partial t} = 0, \tag{5.29}$$
$$\frac{\partial T^2}{\partial t} = 0, \tag{5.30}$$
$$\frac{\partial T^3}{\partial t} = 0, \tag{5.31}$$

respectively. Therefore from (5.22)-(5.27) and (5.29)-(5.31) the components of the conserved vector of the (2+1)-dimensional nonlinear sinh-Gordon equation (5.1) associated with the symmetry X_1 are given by

$$\begin{aligned} T^1 =& c_1 u_t^2 + c_1 u_x^2 + c_2 u_x + c_1 u_y^2 - 2c_1 y u_y \sinh u + u_y \int h_u(x,y,u)dy - a(x,u)u_y + c_3 \\ &+ h(x,y,u), \\ T^2 =& -2c_1 u_t u_x - c_2 u_t + g(x,y), \\ T^3 =& -2c_1 u_t u_y + 2c_1 y u_t \sinh u - u_t \int h_u(x,y,u)dy + a(x,u)u_t - \int g_x(x,y)dy + b(x). \end{aligned}$$

Continuing in the same manner for the remaining symmetries we obtain the components of the conserved vector for equation (5.1) as follows.
In case of X_2 we have

$$\begin{aligned} T^1 =& c_1 u_t^2 + c_1 u_x^2 + c_2 u_x + c_1 u_y^2 + 2c_1 \cosh u + f(y) - g(y,u)t\, u_y - a(y,u)u_y \\ &- t\int g_y(y,u)du - \int a_y(y,u)du + c_4, \\ T^2 =& -2c_1 u_t u_x - c_2 u_t + g(t,y), \\ T^3 =& -2c_1 u_t u_y + g(y,u)t\, u_t + a(y,u)u_t + \int g(y,u)du + c_5. \end{aligned}$$

Considering X_3 we obtained

$$T^1 = c_1 u_t^2 + c_1 u_x^2 + c_2 u_x + c_1 u_y^2 + 2c_1 \cosh u - \alpha(x,u)t\, u_y - a(x,u)u_y + h(x,t) + c_6,$$

$$T^2 = -\,2c_1u_tu_x - c_2u_t - \int h_t(x,t)dx + g(t),$$
$$T^3 = -\,2c_1u_tu_y + \alpha(x,u)t\,u_t + a(x,u)u_t + \int \alpha(x,u)du + b(x),$$

while the symmetry X_4 results in

$$T^1 = c_2u_x,\ T^2 = -c_2u_t,\ T^3 = c_7$$

symmetry X_5 leads us to this conserved vector

$$T^1 = -a(x,u)u_y,\ T^2 = \beta(t^2 - y^2),\ T^3 = a(x,u)u_t.$$

Lastly, we have X_6 from which we obtain

$$T^1 = \phi(x^2 + y^2) + c_8,\ T^2 = g(t)y,\ T^3 = -g(t)x + c_9,$$

where $c_i, i = 1, ..., 9$ are arbitrary constants. One can note that the symmetries X_4, X_5 and X_6 give trivial conserved quantities.

5.2.2 Application of the Noether theorem

In this subsection, we employ the Noether theorem to construct conservation laws of (5.1). A first-order Lagrangian for equation (5.1) is given by

$$L = (1/2)u_t^2 - (1/2)u_x^2 - (1/2)u_y^2 - \cosh u. \tag{5.32}$$

The Noether point symmetries for the above Lagrangian can be obtained by substituting (5.32) into the Noether operator determining equation (1.37), which gives

$$\begin{aligned}
&-\eta\sinh u + u_t\{\eta_t + u_t\eta_u - u_t(\tau_t + u_t\tau_u) - u_x(\xi_t + u_t\xi_u) - u_y(\psi_t + u_t\psi_u)\}\\
&- u_x\{\eta_x + u_x\eta_u - u_t(\tau_x + u_x\tau_u) - u_x(\xi_x + u_x\xi_u) - u_y(\psi_x + u_x\psi_u)\}\\
&- u_y\{\eta_y + u_y\eta_u - u_t(\tau_y + u_y\tau_u) - u_x(\xi_y + u_y\xi_u) - u_y(\psi_y + u_y\psi_u)\}\\
&+ \{(1/2)u_t^2 - (1/2)u_x^2 - (1/2)u_y^2 - \cosh u\}\{\tau_t + u_t\tau_u + \xi_x + u_x\xi_u + \psi_y + u_y\psi_u)\}\\
&= B_t^1 + u_tB_u^1 + B_x^2 + u_xB_u^2 + B_y^3 + u_yB_u^3.
\end{aligned}$$

Splitting the above equation on the derivatives of u, we obtain

$$\tau_u = 0,\ \xi_u = 0,\ \psi_u = 0,\ \xi_t - \tau_x = 0,\ \psi_t - \tau_y = 0,\ \psi_x + \xi_y = 0,$$

$$\xi_x - \tau_t + 2\eta_u + \psi_y = 0,\ \xi_x - 2\eta_u - \tau_t - \psi_y = 0,$$
$$\psi_y - 2\eta_u - \tau_t - \xi_x = 0,\ \eta_t = B^1_u,\ \eta_x = -B^2_u,\ \eta_y = -B^3_u,$$
$$\eta \sinh u + \tau_t \cosh u + \xi_x \cosh u + \psi_y \cosh u = -B^1_t - B^2_x - B^3_y.$$

Solving the above equations yield

$$\tau = d_2 x + d_4 + d_5,\ \xi = -d_1 y + d_2 t + d_3,\ \psi = d_1 x + d_5 t + d_6,\ \eta = 0,$$
$$B^1_t(t,x,y) + B^2_x(t,x,y) + B^3_y(t,x,y) = 0,$$

where d_1, d_2, d_3, d_4, d_5 and d_6 are arbitrary constants and $B^1(t,x,y)$, $B^2(t,x,y)$ and $B^3(t,x,y)$ are arbitrary functions of t, x and y. We can choose $B^1(t,x,y) = B^2(t,x,y) = B^3(t,x,y) = 0$ as they contribute to the trivial part of the conserved vectors. Thus, we get six Noether point symmetries, namely

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = x\partial_t + t\partial_x$$
$$X_5 = y\partial_t + t\partial_y,\ X_6 = -y\partial_x + x\partial_y.$$

Note: It so happens that the Noether symmetries of (5.1) and Lie point symmetries of (5.1) are exactly the same.

The use of Noether theorem, with $X_1 = \partial_t$, gives the conserved vector

$$T^1_1 = -\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \cosh u,\ T^2_1 = u_t u_x,\ T^3_1 = u_t u_y.$$

Using $X_2 = \partial_x$ and applying the Noether theorem, we obtain

$$T^1_2 = -u_t u_x,\ T^2_2 = \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \cosh u,\ T^3_2 = u_x u_y.$$

Conserved vector corresponding to $X_3 = \partial_y$ and employing the Noether theorem is given by

$$T^1_3 = -u_y u_t,\ T^2_3 = u_x u_y,\ T^3_3 = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - \cosh u.$$

The Noether theorem gives the conserved vector

$$T^1_4 = -x(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \cosh u) - t u_x u_t,$$

$$T_4^2 = t(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \cosh u) + xu_xu_t, \ T_4^3 = xu_tu_y + tu_xu_y$$

corresponding to $X_4 = x\partial_t + t\partial_x$.

The conserved vector associated with $X_5 = y\partial_t + t\partial_y$ is given by

$$T_5^1 = y(-\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \cosh u) - tu_tu_y, \ T_5^2 = yu_tu_x + tu_xu_y,$$
$$T_5^3 = t(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - \cosh u) + yu_tu_y.$$

Lastly the Noether theorem gives rise to the conserved vector

$$T_6^1 = yu_tu_x - xu_tu_y, \ T_6^2 = -y(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \cosh u) + xu_xu_y,$$
$$T_6^3 = x(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - \cosh u) - yu_xu_y,$$

for Noether symmetry $X_6 = -y\partial_x + x\partial_y$.

5.2.3 Application of the new conservation theorem

In this subsection we use the new conservation theorem [49] to construct conservation laws for the (2+1)-dimensional nonlinear sinh-Gordon equation (5.1). The adjoint equation of (5.1), by invoking (1.40), is

$$E^*(t, x, u, v, \ldots, u_{xx}, v_{xx}) = \frac{\delta}{\delta u}\left[v\left(u_{tt} - u_{xx} - u_{yy} + \sinh u\right)\right] = 0, \quad (5.33)$$

where $v = v(t, x)$ is a new dependent variable. Thus from (5.33) we have

$$v_{tt} - v_{xx} - v_{yy} + v\cosh u = 0. \quad (5.34)$$

It is clear from the adjoint equation (5.34) that equation (5.1) is not self-adjoint. By recalling (1.45), we obtain the following Lagrangian for the system of equations (5.1) and (5.34):

$$L = v(u_{tt} - u_{xx} - u_{yy} + \sinh u). \quad (5.35)$$

(i) We first consider the Lie point symmetry generator $X_1 = \partial_t$. It can easily be shown from (1.42) that the operator Y_1 is the same as X_1 and hence the Lie characteristic function $W = -u_t$. By using (1.44), the components T^i, $i = 1, 2, 3$, of the conserved vector $T = (T^1, T^2, T^3)$ are given by

$$T_1^1 = v(-u_{xx} - u_{yy} + \sinh u) + u_t v_t, \quad T_1^2 = -u_t v_x + v u_{tx},$$
$$T_1^3 = -u_t v_y + v u_{ty}.$$

Remark. The conserved vector T contains the arbitrary solution v of the adjoint equation (5.34) and hence gives an infinite number of conservation laws. This remark applies to the following five cases where we use the conservation theorem.

(ii) For the symmetry generator $X_2 = \partial_x$, we have $W = -u_x$. Thus, by using (1.44), the symmetry generator X_2 gives rise to the following components of the conserved vector:

$$T_2^1 = v_t u_x - v u_{tx}, \quad T_2^2 = v(u_{tt} - u_{yy} + \sinh u) - v_x u_x,$$
$$T_2^3 = -v_y u_x + v u_{xy}.$$

(iii) For the symmetry generator $X_3 = \partial_y$, we have $W = -u_y$. Using (1.44), the X_3 gives rise to the conserved vector with components

$$T_3^1 = v_t u_y - v u_{ty}, \quad T_3^2 = -v_x u_y + v u_{xy},$$
$$T_3^3 = v(u_{tt} - u_{xx} + \sinh u) - v_y u_y.$$

(iv) The symmetry generator $X_4 = x\partial_t + t\partial_x$ has the Lie characteristic function $W = -x u_t - t u_x$. Thus, invoking (1.44), we obtain the conserved vector T, given by

$$\begin{aligned}
T_4^1 &= xv(-u_{xx} - u_{yy} + \sinh u) + x v_t u_t + t v_t u_x - v u_x - t v u_{tx}, \\
T_4^2 &= tv(u_{tt} - u_{yy} + \sinh u) - x v_x u_t - t v_x u_x + v u_t + x v u_{tx}, \\
T_4^3 &= -v_y x u_t - t v_y u_x + x v u_{ty} + t v u_{xy}.
\end{aligned}$$

(v) The symmetry generator $X_5 = y\partial_t + t\partial_y$ has the Lie characteristic function $W = -y u_t - t u_y$. Using (1.44), we obtain the conserved vector T, given by

$$\begin{aligned}
T_5^1 &= yv(-u_{xx} - u_{yy} + \sinh u) + y v_t u_t + t v_t u_y - v u_y - t v u_{ty}, \\
T_5^2 &= -y v_x u_t - t v_x u_y + y v u_{xt} + t v u_{xy},
\end{aligned}$$

$$T_5^3 = tv(u_{tt} - u_{yy} + \sinh u) - yv_y u_t - tv_y u_y + vu_t + yvu_{ty}.$$

(vi) The symmetry generator $X_6 = -y\partial_x + x\partial_y$ has the Lie characteristic function $W = yu_x - xu_y$. Thus, using (1.44), we obtain the conserved vector T, given by

$$\begin{aligned} T_6^1 &= -yu_x v_t + xu_y v_t + vyu_{xt} - xvu_{yt}, \\ T_6^2 &= -yv(u_{tt} - u_{yy} + \sinh u) + yv_x u_x - xv_x u_y + vu_y + xvu_{xy}, \\ T_6^3 &= xv(u_{tt} - u_{xx} + \sinh u) + yv_y u_x - xv_y u_y - vu_x - yvu_{xy}. \end{aligned}$$

5.3 Concluding remarks

In this chapter the (2+1)-dimensional nonlinear sinh-Gordon equation (5.1) was investigated. Firstly, we obtained exact solutions of (5.1) by using the Lie symmetry analysis along with the simplest equation method and (G'/G)-expansion method. Furthermore, several conserved quantities for equation (5.1) were derived by employing three different techniques; the direct method, the Noether theorem and the new conservation theorem.

Chapter 6

On the solutions and conservation laws for the (3+1)-dimensional nonlinear sinh-Gordon equation

In Chapter 5, we studied (2+1)-dimensional nonlinear sinh-Gordon equation

$$u_{tt} - u_{xx} - u_{yy} + \sinh u = 0. \tag{6.1}$$

In this chapter, we consider the above equation in higher dimension, which is given by [46]

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} + \sinh u = 0 \tag{6.2}$$

and obtain exact solutions of the (3+1)-dimensional nonlinear sinh-Gordon equation (6.2). We also derive conservation laws for (6.2) using two methods, namely Noether theorem and the new theorem by Ibragimov.

This work is new and has been submitted for publication. See [70].

6.1 Symmetry reductions and exact solutions of (6.2)

The symmetry group of (6.2) will be generated by the vector field of the form

$$\begin{aligned} X &= \tau(t,x,y,z,u)\partial_t + \xi(t,x,y,z,u)\partial_x + \varphi(t,x,y,z,u)\partial_y + \phi(t,x,y,z,u)\partial_z \\ &\quad + \eta(t,x,y,z,u)\partial_u. \end{aligned}$$

Applying the second prolongation $X^{[2]}$ to (6.2) we obtain the following overdetermined system of partial differential equations:

$$\begin{aligned} &\xi_u = 0,\ \tau_u = 0,\ \varphi_u = 0,\ \phi_u = 0,\ \eta_{uu} = 0,\ \xi_t - \tau_x = 0,\ \tau_y - \varphi_t = 0,\ \xi_x - \phi_z = 0, \\ &\tau_z - \phi_t = 0,\ \xi_z + \phi_x = 0,\ \phi_z - \varphi_y = 0,\ \tau_{tt} - \tau_{xx} - \tau_{yy} - \tau_{zz} - 2\eta_{tu} = 0, \\ &\xi_y + \varphi_x = 0,\ \varphi_z + \phi_y = 0,\ \tau_t - \phi_z = 0,\ \phi_{tt} - \phi_{xx} - \phi_{yy} - \phi_{zz} + 2\eta_{zu} = 0, \\ &\xi_{tt} - \xi_{xx} - \xi_{yy} - \xi_{zz} + 2\eta_{xu} = 0,\ \varphi_{tt} - \varphi_{xx} - \varphi_{yy} - \varphi_{zz} + 2\eta_{yu} = 0, \\ &\eta\cosh(u) + 2\phi_z\sinh(u) - \eta_u\sinh(u) + \eta_{tt} - \eta_{xx} - \eta_{yy} - \eta_{zz} = 0. \end{aligned}$$

Solving the above equations one obtains the following ten Lie point symmetries:

$$\begin{aligned} &X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = \partial_z,\ X_5 = -z\partial_x + x\partial_z,\ X_6 = z\partial_t + t\partial_z \\ &X_7 = -z\partial_y + y\partial_z,\ X_8 = y\partial_t + t\partial_y,\ X_9 = x\partial_t + t\partial_x,\ X_{10} = -y\partial_x + x\partial_y. \end{aligned}$$

We now use the four translational symmetries X_1, X_2, X_3 and X_4 to transform (6.2) to a partial differential equation depending on three variables. The linear combination of these four symmetries, namely

$$X = X_1 + X_2 + X_3 + X_4$$

gives the following invariants:

$$f = x - t, \quad g = y - t, \quad h = z - t, \quad v = u. \tag{6.3}$$

Taking f, g and h as new independent variables and v as the new dependent variable, equation (6.2) transforms to a nonlinear partial differential equation in three dependent variables

$$2v_{fg} + 2v_{fh} + 2v_{gh} + \sinh(v) = 0. \tag{6.4}$$

Now we will further reduce (6.4) using its symmetries. Equation (6.4) admits the following six symmetries:

$$V_1 = \partial_f,\ V_2 = \partial_g,\ V_3 = \partial_h,\ V_4 = f\partial_f + (f-h)\partial_g - h\partial_h,$$
$$V_5 = (g-h)\partial_f + g\partial_g - h\partial_h,\ V_6 = (g-h)\partial_f + (h-f)\partial_g + (f-g)\partial_h.$$

The symmetry $V = V_1 + V_2 + kV_3$, yields the invariants $r = g - f$, $s = h - kf$ and $w = v$, which gives the group invariant solution $v = w(r, s)$ and consequently using these invariants, equation (6.4) transforms to

$$2kw_{rr} + 2kw_{rs} + 2w_{ss} - \sinh(w) = 0. \tag{6.5}$$

Now equation (6.5) admits the three symmetries

$$\Lambda_1 = \partial_r,\ \Lambda_2 = \partial_s,\ \Lambda_3 = (r - \frac{2s}{k})\partial_r + (2r - s)\partial_s.$$

The symmetry $\Lambda_1 + \Lambda_2$ gives the invariants $\chi = s - r$ and $w = H$. Using these invariants we obtain the group-invariant solution $w = H(\chi)$, where H satisfies the second-order nonlinear ODE

$$2H''(\chi) - \sinh(H) = 0. \tag{6.6}$$

Using the transformation $H(\chi) = \ln(\psi(\chi))$ on (6.6), we get

$$4\psi(\chi)\psi''(\chi) - 4\psi'(\chi)^2 - \psi(\chi)^3 + \psi(\chi) = 0. \tag{6.7}$$

6.1.1 Exact solutions of (6.2) using simplest equation method

We now use the simplest equation method [65,66,71], which was used in the previous chapter, and obtain exact solutions of the nonlinear ODE (6.7). Consequently this will then give us the exact solutions for equation (6.2).

Solutions of (6.2) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of (6.7) take the form

$$\psi(\chi) = A_0 + A_1 G + A_2 G^2. \tag{6.8}$$

Substituting (6.8) and its derivatives into (6.7) and using the Bernoulli equation [69] and thereafter, equating the coefficients of powers of G^i to zero, we obtain an algebraic system of seven equations in terms of A_0, A_1, A_2, namely

$$
\begin{aligned}
A_0^3 - A_0 &= 0,\\
4c^2b^2A_2^2 - A_2^3 &= 0,\\
2c^2a^2A_0A_1 - 3A_0^2A_1 + A_1 &= 0,\\
8c^2b^2A_1A_2 + 4c^2abA_2^2 - 3A_1A_2^2 &= 0,\\
6c^2abA_0A_1 - 3A_0^2A_2 + 8c^2a^2A_0A_2 + A_2 - 3A_0A_1^2 &= 0,\\
10c^2abA_1A_2 - 3A_1^2A_2 + 2c^2b^2A_1^2 + 12c^2b^2A_0A_2 - 3A_0A_2^2 &= 0,\\
4c^2A_0A_2b^2 + 20c^2abA_0A_2 + 2c^2a^2A_1A_2 - 6A_0A_1A_2 + 2c^2abA_1^2 - A_1^3 &= 0.
\end{aligned}
$$

Solving the above system of algebraic equations, with the aid of Maple, one possible solution for A_0, A_1 and A_2 is

$$a = \frac{1}{\sqrt{2}},\ A_0 = 1,\ A_1 = \frac{8b}{\sqrt{2}},\ A_2 = 8b^2.$$

Thus, reverting back to the original variables, a solution of (6.2) is

$$
\begin{aligned}
u(t,x,y,z) = \ln \Bigg(A_0 + aA_1 &\left\{ \frac{\cosh[a(\chi+C)] + \sinh[a(\chi+C)]}{1 - b\cosh[a(\chi+C)] - b\sinh[a(\chi+C)]} \right\}\\
+ a^2A_2 &\left\{ \frac{\cosh[a(\chi+C)] + \sinh[a(\chi+C)]}{1 - b\cosh[a(\chi+C)] - b\sinh[a(\chi+C)]} \right\}^2 \Bigg), \qquad (6.9)
\end{aligned}
$$

where $\chi = (k-1)t + (1-k)x - y + z$ and C is an arbitrary constant of integration.

Solutions of (6.2) using the Riccati equation as the simplest equation

In this case the balancing procedure gives $M = 2$, so the solutions of (6.7) are of the form

$$H(z) = A_0 + A_1G + A_2G^2. \qquad (6.10)$$

Inserting (6.10) into (6.7) and using the Riccati equation [69], we obtain algebraic system of equations in terms of A_0, A_1, A_2 by equating the coefficients powers of G^i

to zero. The resulting algebraic equations are

$$-A_2^3+4c^2a^2A_2^2=0,$$

$$-3A_1A_2^2+8ca^2A_1A_2+4c^2abA_2^2=0,$$

$$10cabA_1A_2+2ca^2A_1^2+12ca^2A_0A_2-3A_1^2A_2-3A_0A_2^2=0,$$

$$2cabA_1^2-A_1^3-6A_0A_1A_2+20abcA_0A_2+4ac\nu A_1A_2+2cb^2A_1A_2-4bc\nu A_2^2 +4a^2cA_0A_1=0,$$

$$16ac\nu A_0A_2-4c\nu^2A_2^2-3A_0A_1^2+A_2+8b^2cA_0A_2+6abcA_0A_1-3A_0^2A_2 -2bc\nu A_1A_2=0,$$

$$-4c\nu^2A_1A_2-2bc\nu A_1^2+12bc\nu A_0A_2+2b^2cA_0A_1-3A_0^2A_1+4ac\nu A_0A_1+A_1=0,$$

$$A_0+2bc\nu A_0A_1-A_0^3+4c\nu^2A_0A_2+2c\nu^2A_1^2=0.$$

Solving the above equations, we get

$$b=\frac{\rho}{8},\ A_0=8a\nu-1,\ A_1=a\,\rho,\ A_2=8a^2,$$

where ρ is any root of $\rho^2-256a\nu+32=0$. Consequently, the solutions of (6.2) are

$$u(t,x,y,z)=\ln\Bigg(A_0+A_1\Bigg\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(\chi+C)\right]\Bigg\} +A_2\Bigg\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(\chi+C)\right]\Bigg\}^2\Bigg) \tag{6.11}$$

and

$$u(t,x,y,z)=\ln\Bigg(A_0+A_1\Bigg\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta\chi\right)+\frac{\operatorname{sech}\left(\frac{\theta\chi}{2}\right)}{C\cosh\left(\frac{\theta\chi}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\Bigg\} +A_2\Bigg\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta\chi\right)+\frac{\operatorname{sech}\left(\frac{\theta\chi}{2}\right)}{C\cosh\left(\frac{\theta\chi}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta\chi}{2}\right)}\Bigg\}^2\Bigg), \tag{6.12}$$

where $\chi=(k-1)t+(1-k)x-y+z$ and C is an arbitrary constant of integration.

6.1.2 Solutions of (6.2) using (G'/G)-expansion method

In this subsection we use the $(G'/G)-$expansion method, which was used in Chapter 5, to solve (6.7) and as a result we obtain the exact solutions of the equation (6.2).

Application of the balancing procedure to the ODE (6.7) yields $M = 2$, so the solutions of (6.7) are of the form

$$\psi(\chi) = \alpha_0 + \alpha_1\left(\frac{G'(\chi)}{G(\chi)}\right) + \alpha_2\left(\frac{G'(\chi)}{G(\chi)}\right)^2. \tag{6.13}$$

Substituting (6.13) into (6.7) and making use of the second-order ODE (5.14) leads to the following overdetermined system of algebraic equations:

$$\begin{aligned}
4\alpha_1\alpha_0\lambda\mu + 8\alpha_2\alpha_0\mu^2 - 4\alpha_1^2\mu^2 - \alpha_0^3 + \alpha_0 &= 0,\\
4\alpha_0\alpha_1\lambda^2 - 4\alpha_1^2\lambda\mu + 24\alpha_0\alpha_2\lambda\mu - 8\alpha_1\alpha_2\mu^2 + 8\alpha_0\alpha_1\mu - 3\alpha_0^2\alpha_1 + \alpha_1 &= 0,\\
16\alpha_0\alpha_2\lambda^2 - 4\alpha_1\alpha_2\lambda\mu + 12\alpha_0\alpha_1\lambda - 8\alpha_2^2\mu^2 + 32\alpha_0\alpha_2\mu - 3\alpha_0\alpha_1^2 - 3\alpha_0^2\alpha_2 + \alpha_2 &= 0,\\
4\alpha_2\alpha_1\lambda^2 - 8\alpha_2^2\lambda\mu + 4\alpha_1^2\lambda + 40\alpha_0\alpha_2\lambda + 8\alpha_2\alpha_1\mu - \alpha_1^3 + 8\alpha_0\alpha_1 - 6\alpha_0\alpha_2\alpha_1 &= 0,\\
20\alpha_2\alpha_1\lambda - 3\alpha_2\alpha_1^2 + 4\alpha_1^2 - 3\alpha_0\alpha_2^2 + 24\alpha_0\alpha_2 &= 0,\\
8\alpha_2^2\lambda - 3\alpha_1\alpha_2^2 + 16\alpha_1\alpha_2 &= 0,\\
8\alpha_2^2 - \alpha_2^3 &= 0.
\end{aligned}$$

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$\alpha_0 = 2\lambda^2,\ \alpha_1 = 8\lambda,\ \alpha_2 = 8,\ \lambda = \sqrt{\frac{1+8\mu}{2}}.$$

Now using the general solution of (5.14) in (6.13), we have the following three types of travelling wave solutions of (6.2):

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solutions

$$u_1(t,x,y,z) = \ln\left[\alpha_0 + \alpha_1\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1\chi) + C_2\cosh(\delta_1\chi)}{C_1\cosh(\delta_1\chi) + C_2\sinh(\delta_1\chi)}\right)\right. \tag{6.14}$$

$$\left. + \alpha_2\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1\chi) + C_2\cosh(\delta_1\chi)}{C_1\cosh(\delta_1\chi) + C_2\sinh(\delta_1 z)}\right)^2\right] \tag{6.15}$$

where $\chi = (k-1)t + (1-k)x - y + z$, $\delta_1 = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solutions

$$u_2(t,x,y,z) = \ln\left[\alpha_0 + \alpha_1\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2\chi) + C_2\cos(\delta_2\chi)}{C_1\cos(\delta_2\chi) + C_2\sin(\delta_2\chi)}\right)\right.$$

$$+\alpha_2\left(-\frac{\lambda}{2}+\delta_2\frac{-C_1\sin(\delta_2\chi)+C_2\cos(\delta_2\chi)}{C_1\cos(\delta_2\chi)+C_2\sin(\delta_2\chi)}\right)^2\Bigg],$$

where $\chi=(k-1)t+(1-k)x-y+z$, $\delta_2=\frac{1}{2}\sqrt{4\mu-\lambda^2}$, C_1 and C_2 are arbitrary constants.

When $\lambda^2-4\mu=0$, we obtain the rational function solutions

$$u_3(t,x,y,z)=\ln\left[\alpha_0+\alpha_1\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2\chi}\right)+\alpha_2\left(-\frac{\lambda}{2}+\frac{C_2}{C_1+C_2\chi}\right)^2\right],$$

where $\chi=(k-1)t+(1-k)x-y+z$, C_1 and C_2 are arbitrary constants.

6.2 Conservation laws of (6.2)

We now construct conservation laws of (6.2) by two different methods, namely, the Noether theorem and the new conservation theorem.

6.2.1 Application of the Noether theorem

In this subsection, we employ the Noether theorem to construct conservation laws of the (3+1)-dimensional nonlinear sinh-Gordon equation (6.2). A first-order Lagrangian for equation (6.2) is given by

$$L=(1/2)u_t^2-(1/2)u_x^2-(1/2)u_y^2-(1/2)u_z^2-\cosh u. \tag{6.16}$$

The Noether point symmetries for the above Lagrangian can be obtained by substituting (6.16) into the Noether operator determining equation (1.37) which gives

$$\begin{aligned}
&u_t\{\eta_t+u_t\eta_u-u_t(\tau_t+u_t\tau_u)-u_x(\xi_t+u_t\xi_u)-u_y(\varphi_t+u_t\varphi_u)-u_z(\phi_t+u_t\phi_u\}\\
&-u_x\{\eta_x+u_x\eta_u-u_t(\tau_x+u_x\tau_u)-u_x(\xi_x+u_x\xi_u)-u_y(\varphi_x+u_x\varphi_u)-u_z(\phi_x+u_x\phi_u)\}\\
&-u_y\{\eta_y+u_y\eta_u-u_t(\tau_y+u_y\tau_u)-u_x(\xi_y+u_y\xi_u)-u_y(\varphi_y+u_y\varphi_u)-u_z(\phi_y+u_y\phi_u\}\\
&-u_z\{\eta_z+u_z\eta_u-u_x(\xi_z+u_z\xi_u)-u_y(\varphi_z+u_z\varphi_u)-u_t(\tau_z+u_z\tau_u)-u_z(\phi_z+u_z\phi_u\}\\
&+\{(1/2)u_t^2-(1/2)u_x^2-(1/2)u_y^2-(1/2)u_z^2-\cosh u\}\{\tau_t+u_t\tau_u+\xi_x+u_x\xi_u+\varphi_y\\
&+u_y\varphi_u+\phi_z+u_z\phi_u\}-\eta\sinh u=B_t^1+u_tB_u^1+B_x^2+u_xB_u^2+B_y^3+u_yB_u^3+B_z^4+u_zB_u^4.
\end{aligned}$$

Splitting the above equation on the derivatives of u, we obtain

$$\tau_u = 0,\ \xi_u = 0,\ \varphi_u = 0,\ \phi_u = 0,\ \xi_t - \tau_x = 0,\ \varphi_t - \tau_y = 0,\ \varphi_x + \xi_y = 0,$$

$$\phi_x + \xi_z = 0,\ \phi_t - \tau_z = 0,\ \phi_y + \varphi_z = 0,\ \xi_x - \tau_t + 2\eta_u + \varphi_y + \phi_z = 0,$$

$$\xi_x - 2\eta_u - \tau_t - \varphi_y - \phi_z = 0,\ \varphi_y - 2\eta_u - \tau_t - \xi_x - \phi_z = 0,$$

$$\phi_z - 2\eta_u - \tau_t - \xi_x - \varphi_y = 0,\ \eta_t = B_u^1,\ \eta_x = -B_u^2,\ \eta_y = -B_u^3,\ \eta_z = -B_u^4,$$

$$\eta \sinh u + \tau_t \cosh u + \xi_x \cosh u + \varphi_y \cosh u + \phi_z \cosh u = -B_t^1 - B_x^2 - B_y^3 - B_z^4.$$

Solving the above equations, we obtain

$$\tau = d_1 x + d_3 y + d_8 z + d_9,\ \xi = d_1 t - d_4 z - d_5 y + d_7,\ \varphi = -d_2 z + d_3 t + d_5 x + d_6,$$

$$\phi = d_2 y + d_4 x + d_8 t + d_{10},\ \eta = 0,$$

$$B_t^1(t,x,y,z) + B_x^2(t,x,y,z) + B_y^3(t,x,y,z) + B_z^4(t,x,y,z) = 0,$$

where $d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9$ and d_{10} are arbitrary constants and $B^1(t,x,y,z)$, $B^2(t,x,y,z)$, $B^3(t,x,y,z)$, and $B^4(t,x,y,z)$ are arbitrary functions of t, x, y and z. We can choose $B^1(t,x,y,z) = B^2(t,x,y,z) = B^3(t,x,y,z) = B^4(t,x,y,z) = 0$ as they contribute to the trivial part of the conserved vectors. Thus, we get ten Noether point symmetries, namely

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = \partial_y,\ X_5 = x\partial_t + t\partial_x,\ X_6 = y\partial_t + t\partial_y,$$

$$X_7 = -y\partial_x + x\partial_y,\ X_8 = -z\partial_y + y\partial_z,\ X_9 = -z\partial_x + x\partial_z,\ X_{10} = z\partial_t + t\partial_z.$$

We note that these Noether symmetries of (6.2) and Lie point symmetries of (6.2) are exactly the same.

The use of the theorem due to Noether, with $X_1 = \partial_t$, gives the conserved vector

$$T_1^1 = -\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2 - \cosh u,\ T_1^2 = u_t u_x,\ T_1^3 = u_t u_y,\ T_1^4 = u_t u_z.$$

Using $X_2 = \partial_x$ and applying the Noether theorem, we obtain

$$T_2^1 = -u_t u_x,\ T_2^2 = \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2 - \cosh u,\ T_2^3 = u_x u_y,\ T_2^4 = u_x u_z.$$

Conserved vector corresponding to $X_3 = \partial_y$ and employing the Noether theorem is given by

$$T_3^1 = -u_y u_t,\ T_3^2 = u_x u_y,\ T_3^3 = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2 - \cosh u,\ T_3^4 = u_y u_z.$$

By using $X_4 = \partial_z$ and employing the Noether theorem, we get

$$T_4^1 = -u_t u_z,\ T_4^2 = u_x u_z,\ T_4^3 = u_y u_z,\ T_4^4 = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2 - \cosh u.$$

The Noether theorem gives us the following conserved vector

$$T_5^1 = y u_t u_x - x u_t u_y,\ T_5^2 = -y\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2 - \cosh u\right) + x u_x u_y,$$
$$T_5^3 = x\left(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2 - \cosh u\right) - y u_x u_y,\ T_5^4 = -y u_x u_z + x u_y u_z,$$

corresponding to $X_5 = -y\partial_x + x\partial_y$.

The Noether theorem gives us the following conserved vector

$$T_6^1 = -x\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2 + \cosh u\right) - t u_t u_x,\ T_6^2 = t\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2\right.$$
$$\left.-\frac{1}{2}u_z^2 - \cosh u\right) + x u_x u_t,\ T_6^3 = x u_t u_y + t u_x u_y,\ T_6^4 = x u_t u_z + t u_x u_z,$$

corresponding to $X_6 = x\partial_t + t\partial_x$.

The Noether theorem gives the following conserved quantity

$$T_7^1 = -y\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2 + \cosh u\right) - t u_t u_y,\ T_7^2 = y u_t u_x + t u_x u_y,$$
$$T_7^3 = t\left(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2 - \cosh u\right) + y u_t u_y,\ T_7^4 = y u_t u_z + t u_y u_z,$$

associated with $X_7 = y\partial_t + t\partial_y$.

The Noether theorem gives rise to the following conserved vector

$$T_8^1 = z u_t u_x - x u_t u_z,\ T_8^2 = -z\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 - \frac{1}{2}u_z^2 - \cosh u\right) + x u_x u_z,$$
$$T_8^3 = -z u_x u_y + x u_y u_z,\ T_8^4 = x\left(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2 - \cosh u\right) - z u_x u_z,$$

for Noether symmetry $X_8 = -z\partial_x + x\partial_z$.

The application of Noether theorem gives us the following conserved vector

$$T_9^1 = -z\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2 + \cosh u\right) - t u_t u_z,\ T_9^2 = z u_t u_x + t u_x u_z,$$

$$T_9^3 = zu_tu_y + tu_yu_z,\ T_9^4 = t\left(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2 - \cosh u\right) + zu_tu_z,$$

for Noether symmetry $X_9 = z\partial_t + t\partial_z$.

Lastly, the Noether theorem gives rise to the following conserved vector

$$T_{10}^1 = zu_tu_y - yu_tu_z,\ T_{10}^2 = -zu_xu_y + yu_xu_z,\ T_{10}^3 = z\left(-\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2 + \cosh u\right) + yu_yu_z,\ T_{10}^4 = y\left(\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{1}{2}u_y^2 + \frac{1}{2}u_z^2 - \cosh u\right) - zu_yu_z,$$

for Noether symmetry $X_{10} = -z\partial_y + y\partial_z$.

6.2.2 Application of the new conservation theorem

In this subsection we use the new conservation theorem given in [49] and construct conservation laws for the (3+1)-dimensional nonlinear sinh-Gordon equation (6.2). The adjoint equation of (6.2), by invoking (1.40), is

$$E^*(t, x, u, v, \ldots, u_{xx}, v_{xx}) = \frac{\delta}{\delta u}\left[v\left(u_{tt} - u_{xx} - u_{yy} - u_{zz} + \sinh u\right)\right] = 0, \quad (6.17)$$

where $v = v(t, x)$ is a new dependent variable. Thus from (6.17) we have

$$v_{tt} - v_{xx} - v_{yy} - v_{zz} + v\cosh u = 0. \quad (6.18)$$

It is clear from the adjoint equation (6.18) that equation (6.2) is not self-adjoint. By recalling (1.45), we obtain the following Lagrangian for the system of equations (6.2) and (6.18):

$$L = v(u_{tt} - u_{xx} - u_{yy} - u_{zz} + \sinh u). \quad (6.19)$$

(i) We first consider the Lie point symmetry generator $X_1 = \partial_t$. It can easily be shown from (1.42) that the operator Y_1 is the same as X_1 and hence the Lie characteristic function $W = -u_t$. Thus, by using (1.44), the components T^i, $i = 1, 2, 3, 4$, of the conserved vector $T = (T^1, T^2, T^3, T^4)$ are given by

$$T_1^1 = v(-u_{xx} - u_{yy} - u_{zz} + \sinh u) + u_tv_t,\ T_1^2 = -u_tv_x + vu_{tx},$$

$$T_1^3 = -u_t v_y + v u_{ty},\ T_1^4 = -u_t v_z + v u_{tz}.$$

Remark. The conserved vector T contains the arbitrary solution v of the adjoint equation (6.18) and hence gives an infinite number of conservation laws.

This remark applies to the following nine cases where we use the conservation theorem.

(ii) For the symmetry generator $X_2 = \partial_x$, we have $W = -u_x$. Thus, by using (1.44), the symmetry generator X_2 gives rise to the following components of the conserved vector:

$$T_2^1 = v_t u_x - v u_{tx},\ T_2^2 = v(u_{tt} - u_{yy} - u_{zz} + \sinh u) - v_x u_x,$$
$$T_2^3 = -v_y u_x + v u_{xy},\ T_2^4 = -u_x v_z + v u_{xz}.$$

(iii) For the symmetry generator $X_3 = \partial_y$, we have $W = -u_y$. Thus, by using (1.44), the symmetry generator X_2 gives rise to the following components of the conserved vector:

$$T_3^1 = v_t u_y - v u_{ty},\ T_3^2 = -v_x u_y + v u_{xy},$$
$$T_3^3 = v(u_{tt} - u_{xx} - u_{zz} + \sinh u) - v_y u_y,\ T_3^4 = -v_z u_y + v u_{yz}.$$

(iv) For the symmetry $X_4 = \partial_z$, we have $W = -u_z$. Thus, by using (1.44), the symmetry generator X_2 gives rise to the conserved quantity

$$T_4^1 = v_t u_z - v u_{tz},\ T_4^2 = -v_x u_z + v u_{xz},$$
$$T_4^3 = -v_y u_z + v u_{yz},\ T_4^4 = v(u_{tt} - u_{xx} - u_{yy} + \sinh u) - v_z u_z.$$

(v) The symmetry generator $X_5 = -z\partial_x + x\partial_z$ has the Lie characteristic function $W = zu_x - xu_z$. Thus, invoking (1.44), we obtain the conserved vector T, given by

$$\begin{aligned}
T_5^1 &= -zv_t u_x + xv_t u_z + zv u_{tx} - xv u_{tz},\\
T_5^2 &= -zv(u_{tt} - u_{yy} - u_{zz} + \sinh u) + zv_x u_x - xv_x u_z + v u_z + xv u_{xz},\\
T_5^3 &= zu_x v_y - xv_y u_z - vz u_{xy} + xv u_{yz},\\
T_5^4 &= xv(u_{tt} - u_{xx} - u_{yy} + \sinh u) + zv_z u_x - xv_z u_z - v u_x - vz u_{xz}.
\end{aligned}$$

(vi) The symmetry generator $X_6 = -z\partial_t + t\partial_z$ has the Lie characteristic function $W = zu_t - tu_z$. Thus, invoking (1.44), we obtain the conserved vector T, given by

$$\begin{aligned}
T_6^1 &= zv(-u_{xx} - u_{yy} - u_{zz} + \sinh u) + zv_tu_t + tv_tu_z - vu_z - tvu_{tz},\\
T_6^2 &= -zv_xu_t - tv_xu_z + zvu_{tx} + tvu_{xz},\\
T_6^3 &= -zu_tv_y - tv_yu_z + zvu_{ty} + tvu_{yz},\\
T_6^4 &= tv(u_{tt} - u_{xx} - u_{yy} + \sinh u) - zv_zu_t - tv_zu_z + vu_t + vzu_{tz}.
\end{aligned}$$

(vii) The generator $X_7 = -z\partial_y + y\partial_z$ has the Lie characteristic function $W = zu_y - yu_z$. Thus, invoking (1.44), we obtain the conserved vector, given by

$$\begin{aligned}
T_7^1 &= -zv_tu_y + yv_tu_z + zvu_{ty} - yvu_{tz},\\
T_7^2 &= zu_yv_x - yv_xu_z - zvu_{xy} + yvu_{xz},\\
T_7^3 &= -zv(u_{tt} - u_{xx} - u_{zz} + \sinh u) + zv_yu_y - yv_yu_z + vu_z + yvu_{yz},\\
T_7^4 &= yv(u_{tt} - u_{xx} - u_{yy} + \sinh u) + zv_zu_y - yv_zu_z - vu_y - zvu_{yz}.
\end{aligned}$$

(viii) The symmetry operator $X_8 = y\partial_t + t\partial_y$ has the Lie characteristic function $W = -yu_t - tu_y$. Thus, using (1.44), we obtain the conserved vector given by

$$\begin{aligned}
T_8^1 &= yv(-u_{xx} - u_{yy} - u_{zz} + \sinh u) + yv_tu_t + tv_tu_y - vu_y - tvu_{ty},\\
T_8^2 &= -yv_xu_t - tv_xu_y + yvu_{xt} + tvu_{xy},\\
T_8^3 &= tv(u_{tt} - u_{xx} - u_{zz} + \sinh u) - yv_yu_t - tv_yu_y + vu_t + yvu_{ty},\\
T_8^4 &= -yu_tv_z - tv_zu_y + yvu_{tz} + tvu_{yz}. \; .
\end{aligned}$$

(ix) The symmetry $X_9 = x\partial_t + t\partial_x$ has the Lie characteristic function $W = -xu_t - tu_x$. Thus, invoking (1.44) we obtain the conserved vector whose components are given by

$$\begin{aligned}
T_9^1 &= xv(-u_{xx} - u_{yy} - u_{zz} + \sinh u) + xv_tu_t + tv_tu_x - vu_x - tvu_{tx},\\
T_9^2 &= tv(u_{tt} - u_{yy} - u_{zz} + \sinh u) - xv_xu_t - tv_xu_x + vu_t + xvu_{tx},\\
T_9^3 &= -xv_yu_t - tv_yu_x + xvu_{ty} + tvu_{xy},\\
T_9^4 &= -xu_zv_z - tv_zu_x + xvu_{tz} + tvu_{xz}.
\end{aligned}$$

(x) Finally, the symmetry operator $X_{10} = -y\partial_x + x\partial_y$ has the Lie characteristic function $W = yu_x - xu_y$. Thus, as before we obtain the conserved vector T whose components are given by

$$
\begin{aligned}
T_{10}^1 &= -yu_xv_t + xu_yv_t + vyu_{xt} - xvu_{yt}, \\
T_{10}^2 &= -yv(u_{tt} - u_{yy} - u_{zz} + \sinh u) + yv_xu_x - xv_xu_y + vu_y + xvu_{xy}, \\
T_{10}^3 &= xv(u_{tt} - u_{xx} - u_{zz} + \sinh u) + yv_yu_x - xv_yu_y - vu_x - yvu_{xy}, \\
T_{10}^4 &= yu_xv_z - xv_zu_y - yvu_{xz} + xvu_{yz}.
\end{aligned}
$$

6.3 Concluding remarks

In this chapter we performed symmetry reduction of the (3+1)-dimensional nonlinear sinh-Gordon equation (6.2). Thereafter the simplest equation method and (G'/G)-expansion method were employed to construct exact solutions of (6.2). We also obtained several conserved quantities for equation (6.2) by applying two different methods, namely the Noether theorem and the new conservation theorem due to Ibragimov.

Chapter 7

Exact solutions and conservation laws of four Boussinesq-type equations

In this chapter, we will study four Boussinesq-type equations, namely the Boussinesq-double sine-Gordon equation, the Boussinesq-double sinh-Gordon equation, the Boussinesq-Liouville type I equation and the Boussinesq-Liouville type II equation given by

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sin u + \frac{3}{2}\sin(2u), \tag{7.1}$$
$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sinh u + \frac{3}{2}\sinh(2u), \tag{7.2}$$
$$u_{tt} - \alpha u_{xx} + u_{xxxx} = e^{u} + \frac{3}{4}e^{2u}, \tag{7.3}$$
$$u_{tt} - \alpha u_{xx} + u_{xxxx} = e^{-u} + \frac{3}{4}e^{-2u}, \tag{7.4}$$

respectively. It is well-known that the Boussinesq equation is a fourth order PDE that includes the physical dispersion term. It appears in integrable quantum field theory, kink dynamics, fluid dynamics and has lots of scientific applications. The aforementioned equations were established by combining the linear Boussinesq equation with the double sine-Gordon, double sinh-Gordon and the Liouville equations. These equations were recently studied in [48] by using simplified Hirota's method and one and two soliton solutions were obtained. Here, we employ the Lie symmetry method together with the simplest equation method and obtain exact solutions of equations (7.1), (7.2), (7.3) and (7.4). In addition to the exact solutions, conservation laws of these equations are derived using the Noether theorem and the multiplier approach.

7.1 Lie point symmetries of (7.1)–(7.4)

The symmetry group of equations (7.1)–(7.4) will be generated by the vector field of the form

$$X = \xi^1(t, x, u)\partial_t + \xi^2(t, x, u)\partial_x + \eta(t, x, u)\partial_u.$$

Applying the fourth prolongation $X^{[4]}$ to (7.1)–(7.4) and solving the resulting overdetermined system of linear partial differential equations of each equation, we obtain the following two Lie point symmetries for each equation:

$$X_1 = \partial_t, \quad X_2 = \partial_x.$$

We now use the above symmetries to transform (7.1)–(7.4) to ordinary differential equations.

7.2 Exact solutions of (7.1)–(7.4)

In this section we construct exact solutions of (7.1)–(7.4).

We now take the linear combination of these two symmetries, namely

$$X = X_1 + cX_2,$$

which gives the following two invariants:

$$z = x - ct, \quad H = u. \tag{7.5}$$

We now treat H as the new dependent variable and z as the new independent variable and transform the PDEs (7.1)–(7.4) to ODEs. We then use the simplest equation method to obtain exact solutions of the ODEs.

7.2.1 Exact solutions of the Boussinesq-double sine-Gordon equation (7.1)

Taking z as the new independent variables and H as the new dependent variable, the substitution of (7.5) into (7.1) gives rise to the second-order nonlinear ODE

$$c^2H''(z) - \alpha H''(z) + H^{(4)}(z) = \sin H + \frac{3}{2}\sin(2H).$$

Then by using the transformation $w = e^{iH}$ we obtain the second-order nonlinear ODE

$$\begin{aligned}
&-4c^2w(z)^3w''(z) + 4c^2w(z)^2w'(z)^2 + 4\alpha w(z)^3w''(z) - 4\alpha w^2(z)w'(z)^2\\
&-4w^{(4)}(z)w(z)^3 + 16w'''(z)w'(z)w(z)^2 - 48w''(z)w'(z)^2w(z) + 12w''(z)^2w(z)^2\\
&+24w'(z)^4 + 2w(z)^5 - 2w(z)^3 + 3w(z)^6 - w(z)^2 = 0. \qquad (7.6)
\end{aligned}$$

Solutions of (7.6) using the Bernoulli equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of (7.6) take the form

$$w(z) = A_0 + A_1G + A_2G^2. \qquad (7.7)$$

Substituting (7.7) and its derivatives into (7.6) and using the Bernoulli equation and thereafter, equating the coefficients of powers of G^i to zero we obtain an algebraic system of thirteen equations in terms of A_0, A_1, A_2, namely

$$\begin{aligned}
&-48\,b^4{A_2}^4 + 3\,{A_2}^6 = 0,\ \ -96\,ab^3{A_2}^4 - 192\,b^4A_1{A_2}^3 + 18\,A_1{A_2}^5 = 0,\\
&3\,{A_0}^6 + 2\,{A_0}^5 - 2\,{A_0}^3 - 3\,{A_0}^2 = 0,\\
&-4\,a^4{A_0}^3A_1 - 4\,a^2c^2{A_0}^3A_1 + 4\,a^2\alpha\,{A_0}^3A_1 + 18\,{A_0}^5A_1 + 10\,{A_0}^4A_1\\
&-6\,{A_0}^2A_1 - 6\,A_0A_1 = 0,\\
&-56\,a^2b^2{A_2}^4 - 384\,ab^3A_1{A_2}^3 - 192\,b^4A_0{A_2}^3 - 288\,b^4{A_1}^2{A_2}^2 - 8\,b^2c^2{A_2}^4
\end{aligned}$$

$$+8\,b^2\alpha\,A_2{}^4+18\,A_0A_2{}^5+45\,A_1{}^2A_2{}^4+2\,A_2{}^5=0,$$

$$-384\,ab^3A_0A_2{}^3-576\,ab^3A_1{}^2A_2{}^2-8\,abc^2A_2{}^4-576\,b^4A_0A_1A_2{}^2-192\,b^4A_1{}^3A_2$$

$$-32\,b^2c^2A_1A_2{}^3+8\,ab\alpha\,A_2{}^4+32\,b^2\alpha\,A_1A_2{}^3+90\,A_0A_1A_2{}^4+60\,A_1{}^3A_2{}^3$$

$$+10\,A_1A_2{}^4-8\,a^3bA_2{}^4-224\,a^2b^2A_1A_2{}^3=0,$$

$$-8\,a^2c^2A_0{}^2A_1{}^2-12\,abc^2A_0{}^3A_1+16\,a^2\alpha\,A_0{}^3A_2+8\,a^2\alpha\,A_0{}^2A_1{}^2+12\,ab\alpha\,A_0{}^3A_1$$

$$+18\,A_0{}^5A_2+45\,A_0{}^4A_1{}^2+10\,A_0{}^4A_2+20\,A_0{}^3A_1{}^2-6\,A_0{}^2A_2-6\,A_0A_1{}^2-6\,A_0A_2$$

$$-64\,a^4A_0{}^3A_2+16\,a^4A_0{}^2A_1{}^2-60\,a^3bA_0{}^3A_1-16\,a^2c^2A_0{}^3A_2-3\,A_1{}^2=0,$$

$$-36\,a^3bA_1A_2{}^3-280\,a^2b^2A_0A_2{}^3-308\,a^2b^2A_1{}^2A_2{}^2-1008\,ab^3A_0A_1A_2{}^2$$

$$-432\,ab^3A_1{}^3A_2-36\,abc^2A_1A_2{}^3-240\,b^4A_0{}^2A_2{}^2-672\,b^4A_0A_1{}^2A_2-24\,b^4A_1{}^4$$

$$+36\,ab\alpha\,A_1A_2{}^3+40\,b^2\alpha\,A_0A_2{}^3+44\,b^2\alpha\,A_1{}^2A_2{}^2+45\,A_0{}^2A_2{}^4+180\,A_0A_1{}^2A_2{}^3$$

$$+45\,A_1{}^4A_2{}^2+10\,A_0A_2{}^4+20\,A_1{}^2A_2{}^3=0,$$

$$+28\,ab\alpha\,A_0{}^2A_1{}^2+8\,b^2\alpha\,A_0{}^3A_1 52\,a^4A_0{}^2A_1A_2-4\,a^4A_0A_1{}^3-520\,a^3bA_0{}^3A_2$$

$$+20\,a^3bA_0{}^2A_1{}^2-200\,a^2b^2A_0{}^3A_1-44\,a^2c^2A_0{}^2A_1A_2-4\,a^2c^2A_0A_1{}^3-40\,abc^2A_0{}^3A_2$$

$$-28\,abc^2A_0{}^2A_1{}^2-8\,b^2c^2A_0{}^3A_1+44\,a^2\alpha\,A_0{}^2A_1A_2+4\,a^2\alpha\,A_0A_1{}^3+40\,ab\alpha\,A_0{}^3A_2$$

$$+90\,A_0{}^4A_1A_2+60\,A_0{}^3A_1{}^3+40\,A_0{}^3A_1A_2+20\,A_0{}^2A_1{}^3-12\,A_0A_1A_2-2\,A_1{}^3$$

$$-6\,A_1A_2=0,$$

$$-4\,a^4A_1A_2{}^3-152\,a^3bA_0A_2{}^3-4\,a^3bA_1{}^2A_2{}^2-296\,a^2b^2A_0A_1A_2{}^2-312\,a^2b^2A_1{}^3A_2$$

$$-96\,ab^3A_0{}^2A_2{}^2-1536\,ab^3A_0A_1{}^2A_2-48\,ab^3A_1{}^4-56\,abc^2A_0A_2{}^3-52\,abc^2A_1{}^2A_2{}^2$$

$$-96\,b^4A_0A_1{}^3-104\,b^2c^2A_0A_1A_2{}^2-24\,b^2c^2A_1{}^3A_2+4\,a^2\alpha\,A_1A_2{}^3+56\,ab\alpha\,A_0A_2{}^3$$

$$+104\,b^2\alpha\,A_0A_1A_2{}^2+24\,b^2\alpha\,A_1{}^3A_2+180\,A_0{}^2A_1A_2{}^3+180\,A_0A_1{}^3A_2{}^2+18\,A_1{}^5A_2$$

$$-4\,a^2c^2A_1A_2{}^3-864\,b^4A_0{}^2A_1A_2+52\,ab\alpha\,A_1{}^2A_2{}^2+20\,A_1{}^3A_2{}^2+40\,A_0A_1A_2{}^3=0,$$

$$256\,a^4A_0{}^2A_2{}^2-32\,a^4A_0A_1{}^2A_2-220\,a^3bA_0{}^2A_1A_2-20\,a^3bA_0A_1{}^3-1320\,a^2b^2A_0{}^3A_2$$

$$-140\,a^2b^2A_0{}^2A_1{}^2-32\,a^2c^2A_0{}^2A_2{}^2-32\,a^2c^2A_0A_1{}^2A_2-240\,ab^3A_0{}^3A_1$$

$$-124\,abc^2A_0{}^2A_1A_2-20\,abc^2A_0A_1{}^3-24\,b^2c^2A_0{}^3A_2-20\,b^2c^2A_0{}^2A_1{}^2+32\,a^2\alpha\,A_0{}^2A_2{}^2$$

$$+32\,a^2\alpha\,A_0A_1{}^2A_2+124\,ab\alpha\,A_0{}^2A_1A_2+20\,ab\alpha\,A_0A_1{}^3+24\,b^2\alpha\,A_0{}^3A_2+20\,b^2\alpha\,A_0{}^2A_1{}^2$$

$$+45\,A_0{}^4A_2{}^2+180\,A_0{}^3A_1{}^2A_2+45\,A_0{}^2A_1{}^4+20\,A_0{}^3A_2{}^2+60\,A_0{}^2A_1{}^2A_2+10\,A_0A_1{}^4$$

$$-6\,A_0A_2{}^2-6\,A_1{}^2A_2-3\,A_2{}^2=0,$$

$$52\,a^4A_0A_1{A_2}^2 - 4\,a^4{A_1}^3A_2 + 872\,a^3b{A_0}^2{A_2}^2 - 304\,a^3bA_0{A_1}^2A_2 - 4\,a^3b{A_1}^4$$
$$-1424\,a^2b^2{A_0}^2A_1A_2 - 112\,a^2b^2A_0{A_1}^3 - 44\,a^2c^2A_0A_1{A_2}^2 - 4\,a^2c^2{A_1}^3A_2 - 6\,A_1{A_2}^2$$
$$-1344\,ab^3{A_0}^3A_2 - 288\,ab^3{A_0}^2{A_1}^2 - 88\,abc^2{A_0}^2{A_2}^2 - 112\,abc^2A_0{A_1}^2A_2 - 4\,abc^2{A_1}^4$$
$$-96\,b^4{A_0}^3A_1 - 80\,b^2c^2{A_0}^2A_1A_2 - 16\,b^2c^2A_0{A_1}^3 + 44\,a^2\alpha\,A_0A_1{A_2}^2 + 4\,a^2\alpha\,{A_1}^3A_2$$
$$+88\,ab\alpha\,{A_0}^2{A_2}^2 + 112\,ab\alpha\,A_0{A_1}^2A_2 + 4\,ab\alpha\,{A_1}^4 + 80\,b^2\alpha\,{A_0}^2A_1A_2 + 16\,b^2\alpha\,A_0{A_1}^3$$
$$+180\,{A_0}^3A_1{A_2}^2 + 180\,{A_0}^2{A_1}^3A_2 + 18\,A_0{A_1}^5 + 60\,{A_0}^2A_1{A_2}^2 + 40\,A_0{A_1}^3A_2 + 2\,{A_1}^5 = 0,$$
$$-64\,a^4A_0{A_2}^3 + 16\,a^4{A_1}^2{A_2}^2 + 188\,a^3bA_0A_1{A_2}^2 - 76\,a^3b{A_1}^3A_2 + 760\,a^2b^2{A_0}^2{A_2}^2$$
$$-1136\,a^2b^2A_0{A_1}^2A_2 - 28\,a^2b^2{A_1}^4 - 16\,a^2c^2A_0{A_2}^3 - 8\,a^2c^2{A_1}^2{A_2}^2 - 2016\,ab^3{A_0}^2A_1A_2$$
$$-192\,ab^3A_0{A_1}^3 - 148\,abc^2A_0A_1{A_2}^2 - 28\,abc^2{A_1}^3A_2 - 480\,b^4{A_0}^3A_2 - 144\,b^4{A_0}^2{A_1}^2$$
$$-56\,b^2c^2{A_0}^2{A_2}^2 - 80\,b^2c^2A_0{A_1}^2A_2 - 4\,b^2c^2{A_1}^4 + 16\,a^2\alpha\,A_0{A_2}^3 + 8\,a^2\alpha\,{A_1}^2{A_2}^2$$
$$+148\,ab\alpha\,A_0A_1{A_2}^2 + 28\,ab\alpha\,{A_1}^3A_2 + 56\,b^2\alpha\,{A_0}^2{A_2}^2 + 80\,b^2\alpha\,A_0{A_1}^2A_2 + 4\,b^2\alpha\,{A_1}^4$$
$$+60\,{A_0}^3{A_2}^3 + 270\,{A_0}^2{A_1}^2{A_2}^2 + 90\,A_0{A_1}^4A_2 + 3\,{A_1}^6 + 20\,{A_0}^2{A_2}^3 + 60\,A_0{A_1}^2{A_2}^2$$
$$+10\,{A_1}^4A_2 - 2\,{A_2}^3 = 0.$$

With the aid of Maple, solving the above system of algebraic equations, one possible solution for A_0, A_1 and A_2 is

$$a = -1, \quad \alpha = c^2 - 3, \quad A_0 = 1, \quad A_1 = -4b, \quad A_2 = 4b^2,$$

Thus, reverting back to the original variables, a solution of (7.1) is

$$\begin{aligned} u(t,x) &= -i\ln\left(A_0 + aA_1\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\right. \\ &\quad \left. +a^2A_2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2\right), \end{aligned} \tag{7.8}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

Solutions of (7.6) using the Riccati equation as the simplest equation

The balancing procedure yields $M = 2$, so the solutions of (7.6) take the form

$$w(z) = A_0 + A_1G + A_2G^2. \tag{7.9}$$

Inserting (7.9) into (7.6) and making use of the Riccati equation, we obtain algebraic system of equations in terms of A_0, A_1, A_2 by equating the coefficients of powers of

G^i to zero. The resulting algebraic equations are

$$-48\,{A_2}^4a^4+3\,{A_2}^6=0,\ \ -96\,{A_2}^4ba^3-192\,A_1{A_2}^3a^4+18\,A_1{A_2}^5=0,$$

$$-8\,c^2{A_2}^4a^2+8\,\alpha\,{A_2}^4a^2+18\,A_0{A_2}^5-64\,{A_2}^4a^3\nu+45\,{A_1}^2{A_2}^4-288\,{A_1}^2{A_2}^2a^4$$
$$-192\,A_0{A_2}^3a^4+2\,{A_2}^5-384\,A_1{A_2}^3ba^3-56\,{A_2}^4b^2a^2=0,$$

$$-256\,A_1{A_2}^3a^3\nu-8\,c^2{A_2}^4ba-8\,{A_2}^4b^3a+60\,{A_1}^3{A_2}^3-576\,A_0A_1{A_2}^2a^4+10\,A_1{A_2}^4$$
$$+90\,A_0A_1{A_2}^4+8\,\alpha\,{A_2}^4ba+32\,\alpha\,A_1{A_2}^3a^2-224\,A_1{A_2}^3b^2a^2-384\,A_0{A_2}^3ba^3$$
$$-32\,c^2A_1{A_2}^3a^2-64\,{A_2}^4ba^2\nu-192\,{A_1}^3A_2a^4-576\,{A_1}^2{A_2}^2ba^3=0,$$

$$-96\,A_2\nu^4{A_1}^2A_0-3\,{A_0}^2-32\,{A_0}^3A_1ba\nu^2+32\,{A_1}^2a\nu^3{A_0}^2-8\,c^2{A_0}^3A_2\nu^2$$
$$+2\,{A_0}^5+4\,\alpha\,{A_0}^3A_1b\nu+48\,{A_2}^2\nu^4{A_0}^2-4\,{A_0}^3A_1b^3\nu+24\,{A_1}^4\nu^4-4\,\alpha\,{A_0}^2{A_1}^2\nu^2$$
$$-64\,{A_0}^3A_2a\nu^3+4\,c^2{A_0}^2{A_1}^2\nu^2+28\,{A_1}^2b^2\nu^2{A_0}^2+8\,\alpha\,{A_0}^3A_2\nu^2-48\,{A_1}^3b\nu^3A_0$$
$$+144\,A_2b\nu^3A_1{A_0}^2-56\,{A_0}^3A_2b^2\nu^2+3\,{A_0}^6-2\,{A_0}^3-4\,c^2{A_0}^3A_1b\nu=0,$$

$$-672\,A_0{A_1}^2A_2a^4+45\,{A_0}^2{A_2}^4-36\,A_1{A_2}^3b^3a-40\,c^2A_0{A_2}^3a^2+20\,{A_1}^2{A_2}^3$$
$$+45\,{A_1}^4{A_2}^2-320\,A_0{A_2}^3a^3\nu+36\,\alpha\,A_1{A_2}^3ba+10\,A_0{A_2}^4-352\,{A_1}^2{A_2}^2a^3\nu$$
$$-240\,{A_0}^2{A_2}^2a^4-432\,{A_1}^3A_2ba^3+40\,\alpha\,A_0{A_2}^3a^2-44\,c^2{A_1}^2{A_2}^2a^2-36\,c^2A_1{A_2}^3ba$$
$$-308\,{A_1}^2{A_2}^2b^2a^2-1008\,A_0A_1{A_2}^2ba^3-24\,{A_1}^4a^4-288\,A_1{A_2}^3ba^2\nu-280\,A_0{A_2}^3b^2a^2$$
$$+44\,\alpha\,{A_1}^2{A_2}^2a^2+180\,A_0{A_1}^2{A_2}^3=0,$$

$$-4\,c^2{A_0}^3A_1b^2+8\,c^2A_0{A_1}^3\nu^2+4\,\alpha\,{A_0}^3A_1b^2-120\,{A_0}^3A_2b^3\nu-64\,{A_0}^3A_1a^2\nu^2-6\,A_0A_1$$
$$-8\,\alpha\,A_0{A_1}^3\nu^2+480\,{A_2}^2b\nu^3{A_0}^2-88\,{A_1}^3b^2\nu^2A_0-32\,{A_1}^3a\nu^3A_0-288\,{A_2}^2\nu^4A_1A_0$$
$$-6\,{A_0}^2A_1+10\,{A_0}^4A_1+18\,{A_0}^5A_1-4\,{A_0}^3A_1b^4+48\,{A_1}^4b\nu^3+96\,A_2\nu^4{A_1}^3-8\,c^2{A_0}^3A_1a\nu$$
$$-24\,c^2{A_0}^3A_2b\nu-4\,c^2{A_0}^2{A_1}^2b\nu-8\,c^2{A_0}^2A_1A_2\nu^2+8\,\alpha\,{A_0}^3A_1a\nu+24\,\alpha\,{A_0}^3A_2b\nu$$
$$+4\,\alpha\,{A_0}^2{A_1}^2b\nu+8\,\alpha\,{A_0}^2A_1A_2\nu^2-88\,{A_0}^3A_1b^2a\nu-480\,{A_0}^3A_2a\nu^2b+112\,{A_0}^2{A_1}^2ba\nu^2$$
$$+376\,{A_0}^2A_1A_2b^2\nu^2+224\,{A_0}^2A_1A_2a\nu^3-384\,A_2b\nu^3{A_1}^2A_0=0,$$

$$-24\,c^2{A_1}^3A_2a^2-4\,c^2A_1{A_2}^3b^2+8\,c^2{A_2}^4b\nu+24\,\alpha\,{A_1}^3A_2a^2+4\,\alpha\,A_1{A_2}^3b^2-8\,\alpha\,{A_2}^4b\nu$$
$$-864\,{A_0}^2A_1A_2a^4-96\,{A_0}^2{A_2}^2ba^3-152\,A_0{A_2}^3b^3a-288\,{A_1}^3A_2a^3\nu-312\,{A_1}^3A_2b^2a^2$$
$$-4\,{A_1}^2{A_2}^2b^3a-64\,A_1{A_2}^3a^2\nu^2+64\,{A_2}^4a\nu^2b+20\,{A_1}^3{A_2}^2+18\,{A_1}^5A_2-96\,A_0{A_1}^3a^4$$
$$-48\,{A_1}^4ba^3-4\,A_1{A_2}^3b^4+8\,{A_2}^4b^3\nu+40\,A_0A_1{A_2}^3+180\,A_0{A_1}^3{A_2}^2+180\,{A_0}^2A_1{A_2}^3$$
$$-104\,c^2A_0A_1{A_2}^2a^2-56\,c^2A_0{A_2}^3ba-52\,c^2{A_1}^2{A_2}^2ba-8\,c^2A_1{A_2}^3a\nu+104\,\alpha\,A_0A_1{A_2}^2a^2$$

$$+56\,\alpha\,A_0{A_2}^3ba+52\,\alpha\,{A_1}^2{A_2}^2ba+8\,\alpha\,A_1{A_2}^3a\nu-1536\,A_0{A_1}^2A_2ba^3-544\,A_0A_1{A_2}^2a^3\nu$$

$$-296\,A_0A_1{A_2}^2b^2a^2-736\,A_0{A_2}^3ba^2\nu-272\,{A_1}^2{A_2}^2ba^2\nu-88\,A_1{A_2}^3b^2a\nu=0,$$

$$-16\,c^2{A_0}^3A_2b^2-8\,c^2{A_0}^2{A_1}^2b^2-8\,c^2{A_0}^2{A_2}^2\nu^2+16\,\alpha\,{A_0}^3A_2b^2+8\,\alpha\,{A_0}^2{A_1}^2b^2$$

$$+8\,\alpha\,{A_0}^2{A_2}^2\nu^2-544\,{A_0}^3A_2a^2\nu^2-60\,{A_0}^3A_1b^3a+16\,{A_0}^2{A_1}^2a^2\nu^2$$

$$+1096\,{A_0}^2{A_2}^2b^2\nu^2+704\,{A_0}^2{A_2}^2a\nu^3-44\,A_0{A_1}^3b^3\nu+192\,A_2b\nu^3{A_1}^3-6\,A_0A_2$$

$$-6\,{A_0}^2A_2-6\,A_0{A_1}^2+10\,{A_0}^4A_2+20\,{A_0}^3{A_1}^2+18\,{A_0}^5A_2+45\,{A_0}^4{A_1}^2$$

$$-52\,c^2{A_0}^2A_1A_2b\nu+592\,{A_0}^2A_1A_2a\nu^2b+52\,\alpha\,{A_0}^2A_1A_2b\nu+4\,c^2{A_1}^4\nu^2$$

$$-64\,{A_0}^3A_2b^4+16\,{A_0}^2{A_1}^2b^4-4\,\alpha\,{A_1}^4\nu^2+28\,{A_1}^4b^2\nu^2+32\,{A_1}^4a\nu^3+144\,{A_2}^2\nu^4{A_1}^2$$

$$-288\,{A_2}^3\nu^4A_0-12\,c^2{A_0}^3A_1ba-32\,c^2{A_0}^3A_2a\nu-16\,c^2{A_0}^2{A_1}^2a\nu+4\,c^2A_0{A_1}^3b\nu$$

$$+16\,c^2A_0{A_1}^2A_2\nu^2+12\,\alpha\,{A_0}^3A_1ba+32\,\alpha\,{A_0}^3A_2a\nu+16\,\alpha\,{A_0}^2{A_1}^2a\nu-4\,\alpha\,A_0{A_1}^3b\nu$$

$$-16\,\alpha\,A_0{A_1}^2A_2\nu^2-240\,{A_0}^3A_1ba^2\nu-928\,{A_0}^3A_2b^2a\nu+284\,{A_0}^2A_1A_2b^3\nu$$

$$+112\,{A_0}^2{A_1}^2b^2a\nu-112\,A_0{A_1}^3ba\nu^2-464\,A_0{A_1}^2A_2b^2\nu^2-256\,A_0{A_1}^2A_2a\nu^3$$

$$-864\,{A_2}^2b\nu^3A_1A_0-3\,{A_1}^2=0,$$

$$3\,{A_1}^6+320\,{A_0}^2{A_2}^2a^3\nu+760\,{A_0}^2{A_2}^2b^2a^2+112\,{A_1}^2{A_2}^2b^2a\nu+16\,A_1{A_2}^3a\nu^2b$$

$$-192\,A_0{A_1}^3ba^3-544\,A_0{A_2}^3a^2\nu^2-76\,{A_1}^3A_2b^3a+16\,{A_1}^2{A_2}^2a^2\nu^2-28\,A_1{A_2}^3b^3\nu$$

$$-56\,c^2{A_0}^2{A_2}^2a^2-16\,c^2A_0{A_2}^3b^2-8\,c^2{A_1}^2{A_2}^2b^2+56\,\alpha\,{A_0}^2{A_2}^2a^2+16\,\alpha\,A_0{A_2}^3b^2$$

$$+8\,\alpha\,{A_1}^2{A_2}^2b^2+20\,{A_0}^2{A_2}^3+10\,{A_1}^4A_2+60\,{A_0}^3{A_2}^3-148\,c^2A_0A_1{A_2}^2ba$$

$$+148\,\alpha\,A_0A_1{A_2}^2ba-176\,A_0A_1{A_2}^2ba^2\nu+4\,\alpha\,{A_1}^4a^2-4\,c^2{A_1}^4a^2-8\,\alpha\,{A_2}^4\nu^2$$

$$+8\,c^2{A_2}^4\nu^2-480\,{A_0}^3A_2a^4-144\,{A_0}^2{A_1}^2a^4-64\,A_0{A_2}^3b^4-28\,{A_1}^4b^2a^2-32\,{A_1}^4a^3\nu$$

$$+16\,{A_1}^2{A_2}^2b^4+56\,{A_2}^4b^2\nu^2+64\,{A_2}^4a\nu^3+60\,A_0{A_1}^2{A_2}^2+90\,A_0{A_1}^4A_2$$

$$+270\,{A_0}^2{A_1}^2{A_2}^2-80\,c^2A_0{A_1}^2A_2a^2-32\,c^2A_0{A_2}^3a\nu-28\,c^2{A_1}^3A_2ba-2\,{A_2}^3$$

$$-16\,c^2{A_1}^2{A_2}^2a\nu+20\,c^2A_1{A_2}^3b\nu+80\,\alpha\,A_0{A_1}^2A_2a^2+32\,\alpha\,A_0{A_2}^3a\nu+28\,\alpha\,{A_1}^3A_2ba$$

$$+16\,\alpha\,{A_1}^2{A_2}^2a\nu-20\,\alpha\,A_1{A_2}^3b\nu-2016\,{A_0}^2A_1A_2ba^3-1024\,A_0{A_1}^2A_2a^3\nu$$

$$-1136\,A_0{A_1}^2A_2b^2a^2+188\,A_0A_1{A_2}^2b^3a-928\,A_0{A_2}^3b^2a\nu-368\,{A_1}^3A_2ba^2\nu=0,$$

$$-8\,c^2{A_0}^3A_1a^2-4\,c^2A_0{A_1}^3b^2+4\,c^2{A_1}^4b\nu+16\,c^2{A_1}^3A_2\nu^2+8\,\alpha\,{A_0}^3A_1a^2$$

$$+4\,\alpha\,A_0{A_1}^3b^2-4\,\alpha\,{A_1}^4b\nu-16\,\alpha\,{A_1}^3A_2\nu^2+88\,\alpha\,A_0A_1{A_2}^2a\nu-6\,A_1{A_2}^2$$

$$-200\,{A_0}^3A_1b^2a^2-160\,{A_0}^3A_1a^3\nu-520\,{A_0}^3A_2b^3a+20\,{A_0}^2{A_1}^2b^3a+52\,{A_0}^2A_1A_2b^4$$

$$+920\,{A_0}^2{A_2}^2b^3\nu - 64\,A_0{A_1}^3a^2\nu^2 + 32\,{A_1}^4ba\nu^2 + 112\,{A_1}^3A_2b^2\nu^2 + 128\,{A_1}^3A_2a\nu^3$$

$$+288\,{A_2}^2b\nu^3{A_1}^2 - 960\,{A_2}^3b\nu^3A_0 - 6\,A_1A_2 + 20\,{A_0}^2{A_1}^3 + 60\,{A_0}^3{A_1}^3$$

$$-88\,c^2{A_0}^2A_1A_2a\nu - 16\,c^2A_0{A_1}^2A_2b\nu + 88\,\alpha\,{A_0}^2A_1A_2a\nu + 16\,\alpha\,A_0{A_1}^2A_2b\nu$$

$$-704\,A_0{A_1}^2A_2a\nu^2b + 184\,{A_0}^2A_1A_2b^2a\nu - 12\,A_0A_1A_2 - 4\,A_0{A_1}^3b^4 + 4\,{A_1}^4b^3\nu$$

$$+96\,{A_2}^3\nu^4A_1 + 40\,{A_0}^3A_1A_2 + 90\,{A_0}^4A_1A_2 - 40\,c^2{A_0}^3A_2ba - 28\,c^2{A_0}^2{A_1}^2ba$$

$$-44\,c^2{A_0}^2A_1A_2b^2 - 40\,c^2{A_0}^2{A_2}^2b\nu - 8\,c^2A_0{A_1}^3a\nu + 16\,c^2A_0A_1{A_2}^2\nu^2 + 40\,\alpha\,{A_0}^3A_2ba$$

$$+28\,\alpha\,{A_0}^2{A_1}^2ba + 44\,\alpha\,{A_0}^2A_1A_2b^2 + 40\,\alpha\,{A_0}^2{A_2}^2b\nu + 8\,\alpha\,A_0{A_1}^3a\nu - 16\,\alpha\,A_0A_1{A_2}^2\nu^2$$

$$-1760\,{A_0}^3A_2ba^2\nu - 128\,{A_0}^2A_1A_2a^2\nu^2 - 80\,{A_0}^2{A_1}^2ba^2\nu + 2560\,{A_0}^2{A_2}^2a\nu^2b$$

$$-208\,A_0{A_1}^2A_2b^3\nu - 88\,A_0{A_1}^3b^2a\nu - 752\,A_0A_1{A_2}^2b^2\nu^2 - 448\,A_0A_1{A_2}^2a\nu^3 - 2\,{A_1}^3 = 0,$$

$$2{A_1}^5 - 16\,c^2A_0{A_1}^3a^2 - 4\,c^2{A_1}^4ba - 4\,c^2{A_1}^3A_2b^2 + 24\,c^2A_1{A_2}^3\nu^2 + 16\,\alpha\,A_0{A_1}^3a^2$$

$$+4\,\alpha\,{A_1}^4ba + 4\,\alpha\,{A_1}^3A_2b^2 - 24\,\alpha\,A_1{A_2}^3\nu^2 - 1344\,{A_0}^3A_2ba^3 - 288\,{A_0}^2{A_1}^2ba^3$$

$$+872\,{A_0}^2{A_2}^2b^3a - 112\,A_0{A_1}^3b^2a^2 - 128\,A_0{A_1}^3a^3\nu + 52\,A_0A_1{A_2}^2b^4 - 488\,A_0{A_2}^3b^3\nu$$

$$-64\,{A_1}^3A_2a^2\nu^2 - 32\,{A_1}^4ba^2\nu + 68\,{A_1}^2{A_2}^2b^3\nu + 24\,A_1{A_2}^3b^2\nu^2 + 96\,A_1{A_2}^3a\nu^3$$

$$+18\,A_0{A_1}^5 - 112\,c^2A_0{A_1}^2A_2ba - 88\,c^2A_0A_1{A_2}^2a\nu + 112\,\alpha\,A_0{A_1}^2A_2ba$$

$$-1472\,A_0{A_1}^2A_2ba^2\nu + 184\,A_0A_1{A_2}^2b^2a\nu - 96\,{A_0}^3A_1a^4 - 4\,{A_1}^4b^3a - 4\,{A_1}^3A_2b^4$$

$$+96\,{A_2}^4b\nu^3 + 40\,A_0{A_1}^3A_2 + 60\,{A_0}^2A_1{A_2}^2 + 180\,{A_0}^2{A_1}^3A_2 + 180\,{A_0}^3A_1{A_2}^2$$

$$-80\,c^2{A_0}^2A_1A_2a^2 - 88\,c^2{A_0}^2{A_2}^2ba - 44\,c^2A_0A_1{A_2}^2b^2 - 8\,c^2A_0{A_2}^3b\nu - 8\,c^2{A_1}^3A_2a\nu$$

$$+20\,c^2{A_1}^2{A_2}^2b\nu + 80\,\alpha\,{A_0}^2A_1A_2a^2 + 88\,\alpha\,{A_0}^2{A_2}^2ba + 44\,\alpha\,A_0A_1{A_2}^2b^2 + 8\,\alpha\,A_0{A_2}^3b\nu$$

$$+8\,\alpha\,{A_1}^3A_2a\nu - 20\,\alpha\,{A_1}^2{A_2}^2b\nu - 1216\,{A_0}^2A_1A_2a^3\nu - 1424\,{A_0}^2A_1A_2b^2a^2$$

$$+2176\,{A_0}^2{A_2}^2ba^2\nu - 304\,A_0{A_1}^2A_2b^3a - 128\,A_0A_1{A_2}^2a^2\nu^2 - 1504\,A_0{A_2}^3a\nu^2b$$

$$-88\,{A_1}^3A_2b^2a\nu + 304\,{A_1}^2{A_2}^2a\nu^2b = 0,$$

$$-24\,c^2{A_0}^3A_2a^2 - 20\,c^2{A_0}^2{A_1}^2a^2 - 32\,c^2{A_0}^2{A_2}^2b^2 + 28\,\alpha\,A_0A_1{A_2}^2b\nu - 6\,{A_1}^2A_2$$

$$+8\,c^2A_0{A_2}^3\nu^2 + 28\,c^2{A_1}^2{A_2}^2\nu^2 + 24\,\alpha\,{A_0}^3A_2a^2 + 20\,\alpha\,{A_0}^2{A_1}^2a^2 + 32\,\alpha\,{A_0}^2{A_2}^2b^2$$

$$-8\,\alpha\,A_0{A_2}^3\nu^2 - 28\,\alpha\,{A_1}^2{A_2}^2\nu^2 - 960\,{A_0}^3A_2a^3\nu - 240\,{A_0}^3A_1ba^3 - 1320\,{A_0}^3A_2b^2a^2$$

$$-140\,{A_0}^2{A_1}^2b^2a^2 - 160\,{A_0}^2{A_1}^2a^3\nu + 1216\,{A_0}^2{A_2}^2a^2\nu^2 - 20\,A_0{A_1}^3b^3a - 32\,A_0{A_1}^2A_2b^4$$

$$-1096\,A_0{A_2}^3b^2\nu^2 - 704\,A_0{A_2}^3a\nu^3 + 12\,{A_1}^3A_2b^3\nu + 196\,{A_1}^2{A_2}^2b^2\nu^2 + 224\,{A_1}^2{A_2}^2a\nu^3$$

$$+144\,{A_2}^3b\nu^3A_1 - 6\,A_0{A_2}^2 + 48\,{A_2}^4\nu^4 + 10\,A_0{A_1}^4 + 20\,{A_0}^3{A_2}^2 + 45\,{A_0}^2{A_1}^4$$

$$+45\,{A_0}^4{A_2}^2 - 64\,c^2A_0{A_1}^2A_2a\nu - 28\,c^2A_0A_1{A_2}^2b\nu + 64\,\alpha\,A_0{A_1}^2A_2a\nu - 3\,{A_2}^2$$
$$-1280\,{A_0}^2A_1A_2ba^2\nu - 704\,A_0{A_1}^2A_2b^2a\nu - 512\,A_0A_1{A_2}^2a\nu^2b - 124\,c^2{A_0}^2A_1A_2ba$$
$$+124\,\alpha\,{A_0}^2A_1A_2ba + 256\,{A_0}^2{A_2}^2b^4 + 60\,{A_0}^2{A_1}^2A_2 + 180\,{A_0}^3{A_1}^2A_2 - 64\,c^2{A_0}^2{A_2}^2a\nu$$
$$-20\,c^2A_0{A_1}^3ba - 32\,c^2A_0{A_1}^2A_2b^2 + 12\,c^2{A_1}^3A_2b\nu + 64\,\alpha\,{A_0}^2{A_2}^2a\nu + 20\,\alpha\,A_0{A_1}^3ba$$
$$+32\,\alpha\,A_0{A_1}^2A_2b^2 - 12\,\alpha\,{A_1}^3A_2b\nu - 220\,{A_0}^2A_1A_2b^3a + 2752\,{A_0}^2{A_2}^2b^2a\nu$$
$$-512\,A_0{A_1}^2A_2a^2\nu^2 - 160\,A_0{A_1}^3ba^2\nu - 124\,A_0A_1{A_2}^2b^3\nu + 96\,{A_1}^3A_2a\nu^2b = 0.$$

Solving this algebraic system of equations we get

$$b = \Upsilon, \quad \nu = \frac{A_0 - 1}{4a}, \quad \alpha = c^2 - 3, \quad A_1 = 4a\Upsilon, \quad A_2 = 4a^2,$$

where Υ is any root of $\Upsilon^2 - A_0 = 0$, and consequently, the solutions of (7.1) are

$$\begin{aligned} u(t,x) &= -i\ln\left(A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\right. \\ &\quad \left. + A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2\right) \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} u(t,x) &= -i\ln\left(A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} + \right. \\ &\quad \left. A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2\right), \end{aligned} \tag{7.11}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

7.2.2 Exact solutions of the Boussinesq-double sinh-Gordon equation (7.2)

We next study the Boussinesq-double sinh-Gordon equation

$$u_{tt} + \alpha u_{xx} + u_{xxxx} = \sinh u + \frac{3}{2}\sinh 2u.$$

Taking z as a new independent variables and H as a the new dependent variable, the substitution of (7.5) into (7.2) give rise to a second-order nonlinear ODE

$$c^2H''(z) - \alpha H''(z) + H^{(4)}(z) = \sinh H + \frac{3}{2}\sinh(2H).$$

Then by using the transformation $w = e^H$, the above equation trasforms to

$$\begin{aligned}
&-4c^2w(z)^3w''(z) + 4c^2w(z)^2w'(z)^2 + 4\alpha w(z)^3w''(z) - 4\alpha w^2(z)w'(z)^2 \\
&-4w^{(4)}(z)w(z)^3 + 16w'''(z)w'(z)w(z)^2 - 48w''(z)w'(z)^2w(z) + 12w''(z)^2w(z)^2 \\
&+24w'(z)^4 + 2w(z)^5 - 2w(z)^3 + 3w(z)^6 - 3w(z)^2 = 0. \qquad (7.12)
\end{aligned}$$

Solutions of (7.12) using the Bernoulli equation as the simplest equation

Following the above procedure, we obtain the values of A_0, A_1 and A_2 as

$$a = -1, \quad \alpha = c^2 - 3, \quad A_0 = 1, \quad A_1 = -4b, \quad A_2 = 4b^2.$$

Thus, reverting back to the original variables, a solution of (7.2) is

$$\begin{aligned}
u(t,x) &= \ln\Bigg(A_0 + aA_1\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\} \\
&\quad + a^2A_2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2\Bigg), \qquad (7.13)
\end{aligned}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

Solutions of (7.12) using the Riccati equation as the simplest equation

Similarly, by using the Riccati equation as the simplest equation we obtain

$$b = \Upsilon, \quad \nu = \frac{A_0 - 1}{4a}, \quad \alpha = c^2 - 3, \quad A_1 = 4a\Upsilon, \quad A_2 = 4a^2,$$

where Υ is any root of $\Upsilon^2 - A_0 = 0$. Consequently, the solutions of (7.2) are

$$\begin{aligned}
u(t,x) &= \ln\Bigg(A_0 + A_1\left\{-\frac{b}{2a} -. \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\} \\
&\quad + A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2\Bigg) \qquad (7.14)
\end{aligned}$$

and

$$\begin{aligned}
u(t,x) &= \ln\Bigg(A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\} + \\
&\quad A_2\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right) + \frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right) - \frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2\Bigg), \qquad (7.15)
\end{aligned}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

7.2.3 Exact solutions of the Boussinesq-Liouville type I equation (7.3)

We next study the Boussinesq-Liouville type I equation. The substitution of (7.5) into (7.3) gives rise to a second-order nonlinear ODE

$$c^2H''(z) - \alpha H''(z) + H^{(4)}(z) = e^H + \frac{3}{4}e^{2H}.$$

Then the transformation $w = e^H$ transforms the above equation to

$$\begin{aligned}
&-4c^2w(z)^3w''(z) + 4c^2w(z)^2w'(z)^2 + 4\alpha w(z)^3w''(z) - 4\alpha w^2(z)w'(z)^2 \\
&-4w^{(4)}(z)w(z)^3 + 16w'''(z)w'(z)w(z)^2 - 48w''(z)w'(z)^2w(z) + 12w''(z)^2w(z)^2 \\
&+24w'(z)^4 + 4w(z)^5 + 3w(z)^6 = 0. \qquad (7.16)
\end{aligned}$$

Solutions of (7.16) using the Bernoulli equation as the simplest equation

Solving (7.16) by using the Bernoulli equation as the simplest equation , one possible solution for A_0, A_1 and A_2 is

$$\alpha = \frac{A_1^2 + 16c^2b^2 - 32b^2}{16b^2}, \quad A_0 = 0, \quad A_1 = 4ab, \quad A_2 = 4b^2.$$

Thus, reverting back to the original variables, a solution of (7.3) is

$$\begin{aligned}
u(t,x) &= \ln\left(aA_1\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}\right. \\
&\left. + a^2A_2\left\{\frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b\cosh[a(z+C)] - b\sinh[a(z+C)]}\right\}^2\right), \qquad (7.17)
\end{aligned}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

Solutions of (7.16) using the Riccati equation as the simplest equation

By using the Riccati equation as the simplest equation, we obtain

$$\alpha = \frac{32a^2 + 16A_0a^2 + 16a^2c^2 + A_1^2}{16a^2}, \quad A_0 = -4a\nu, \quad A_1 = -4ab, \quad A_2 = -4a^2.$$

Thus, the solutions of (7.3) are

$$u(t,x) = \ln\left(A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}\right.$$

$$+A_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(z+C)\right]\right\}^2\Bigg) \qquad (7.18)$$

and

$$\begin{aligned} u(t,x) &= \ln\left(A_0 + A_1\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}+\right. \\ & \left. A_2\left\{-\frac{b}{2a}-\frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta z\right)+\frac{\operatorname{sech}\left(\frac{\theta z}{2}\right)}{C\cosh\left(\frac{\theta z}{2}\right)-\frac{2a}{\theta}\sinh\left(\frac{\theta z}{2}\right)}\right\}^2\right), \qquad (7.19) \end{aligned}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

7.2.4 Exact solutions of the Boussinesq-Liouville type II equation (7.4)

Substitution of (7.5) into (7.4) gives rise to a second-order nonlinear ODE

$$c^2H''(z) - \alpha H''(z) + H^{(4)}(z) = e^{-H} + \frac{3}{4}e^{-2H}.$$

The transformation $w = e^{-H}$ transforms the above equation to

$$\begin{aligned} & 4c^2w(z)^3w''(z) - 4c^2w(z)^2w'(z)^2 - 4\alpha w(z)^3w''(z) + 4\alpha w^2(z)w'(z)^2 \\ & +4w^{(4)}(z)w(z)^3 - 16w'''(z)w'(z)w(z)^2 + 48w''(z)w'(z)^2w(z) - 12w''(z)^2w(z)^2 \\ & -24w'(z)^4 + 4w(z)^5 + 3w(z)^6 = 0. \qquad (7.20) \end{aligned}$$

Solutions of (7.20) using the Bernoulli equation as the simplest equation
Following the above procedure, and using the Bernoulli equation, we get the values A_0, A_1 and A_2 as

$$b = \frac{1}{2}\varrho, \quad \alpha = \frac{1}{4A_2^2}\left[\varrho^2A_1^2 - 8\varrho^2A_2 + 4c^2A_2^2\right], \quad A_0 = 0, \quad A_1 = \frac{2aA_2}{\varrho},$$

where ϱ is any root $\varrho^4 + A_2^2 = 0$. Thus, reverting back to the original variables, a solution of (7.4) is

$$\begin{aligned} u(t,x) &= \ln\left(A_0 + aA_1\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}\right. \\ & \left. +a^2A_2\left\{\frac{\cosh[a(z+C)]+\sinh[a(z+C)]}{1-b\cosh[a(z+C)]-b\sinh[a(z+C)]}\right\}^2\right), \qquad (7.21) \end{aligned}$$

where $z = x - ct$ and C is an arbitrary constant of integration.

Solutions of (7.20) using the Riccati equation as the simplest equation

When the Riccati equation is used as the simplest equation, we get only the trivial solution of (7.20).

7.3 Conservation laws of (7.1)–(7.4)

In this section we construct conservation laws of (7.1)–(7.4) by using two approaches, namely the Noether theorem and multiplier method.

7.3.1 Conservation laws for the Boussinesq-double sine-Gordon equation (7.1)

Application of Noether theorem

In this subsection we employ the Noether theorem [41] to construct conservation laws of the Boussinesq-double sine-Gordon equation (7.1). A second-order Lagrangian for equation (7.1) is given by

$$L = -\frac{1}{2}u_t^2 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_{xx}^2 + \cos u + \frac{3}{4}\cos(2u). \tag{7.22}$$

The Noether point symmetries for the above Lagrangian can be obtained by substituting (7.22) into the Noether operator determining equation (1.37), which gives

$$\begin{aligned}
&-\eta\sin u - (3/2)\sin(2u) - u_t\{\eta_t + u_t\eta_u - u_t(\tau_t + u_t\tau_u) - u_x(\xi_t + u_t\xi_u)\}\\
&+\alpha u_x\{\eta_x + u_x\eta_u - u_t(\tau_x + u_x\tau_u) - u_x(\xi_x + u_x\xi_u)\}\\
&+u_{xx}\{\eta_{xx} + u_{xx}\eta_u + 2u_x\eta_{xu} + u_x^2\eta_{uu} - u_t\tau_{xx} - 2u_xu_{tx}\tau_{xu} - 2u_{tx}\tau_x - u_tu_x^2\tau_{uu}\\
&-u_tu_{xx}\tau_u - 2u_{tx}u_x\tau_u - 2u_{xx}\xi_x - u_x^2\xi_{xu} - 3u_xu_{xx}\xi_u - u_x\xi_{xx} - u_x^3\xi_{uu}\} +\\
&\{(\alpha/2)u_x^2 - (1/2)u_t^2 + (1/2)u_{xx}^2 + \cos u + (3/4)\cos(2u)\}\{\tau_t + u_t\tau_u + \xi_x + u_x\xi_u\}\\
&= B_t^1 + u_tB_u^1 + B_x^2 + u_xB_u^2.
\end{aligned}$$

By splitting the above equation on the derivatives of u, we obtain

$$\tau_u = 0,\ \xi_u = 0,\ \tau_x = 0,\ \eta_{xx} = 0,\ \eta_{uu} = 0,\ -\xi_t + \alpha\tau_x = 0,\ 2\eta_{xu} - \xi_{xx} = 0,$$

$$\xi_x - \tau_t - 2\eta_u = 0,\ 2\eta_u - \tau_t + \xi_x = 0,\ 2\eta_u + \tau_t - 3\xi_x = 0,\ -\eta_t = B^1_u,\ \alpha\eta_x = B^2_u,$$

$$-\eta\sin u - (3/2)\,\eta\,\sin(2u) + \cos u\,\tau_t + (3/4)\cos(2u)\,\tau_t - \cos u\,\xi_x$$

$$-(3/4)\cos(2u)\,\xi_x = B^1_t + B^2_x.$$

Solving the above system of equations, we get

$$\tau = c_1,\ \xi = c_2,\ \eta = 0,$$

$$B^1_t(t,x) + B^2_x(t,x) = 0,$$

where c_1 and c_2 are arbitrary constants and $B^1(t,x)$ and $B^2(t,x)$ are arbitrary functions of t and x. We can take $B^1(t,x) = B^2(t,x) = 0$ since they contribute to the trivial part of the conserved vectors. Thus, we get the following Noether point symmetries:

$$X_1 = \partial_t, \quad X_2 = \partial_x.$$

The use of the theorem due to Noether with $X_1 = \partial_t$, gives the conserved vector

$$T_1^1 = \frac{1}{2}u_t^2 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_{xx}^2 + \cos u + \frac{3}{4}\cos(2u), \quad T_1^2 = -\alpha u_t u_x + u_t u_{xxx} - u_{xt}u_{xx}.$$

Using $X_2 = \partial_x$ and applying the Noether theorem, we obtain

$$T_2^1 = u_t u_x, \quad T_2^2 = -\frac{1}{2}u_t^2 - \frac{\alpha}{2}u_x^2 - \frac{1}{2}u_{xx}^2 + u_x u_{xxx} + \cos u + \frac{3}{4}\cos(2u).$$

Application of the multiplier method

In this subsection we utilize the multiplier method [50] to obtain conservation laws of (7.1). After some straightforward but lengthy calculations we obtain the first-order multiplier for (7.1), viz.,

$$\Lambda = \Lambda(t,x,u,u_t,u_x) = C_1 u_t + C_2 u_x,$$

where $C_i, i = 1,2$ are arbitrary constants. The above multiplier yields the following two local conserved vectors of (7.1):

$$T_1^1 \ = \ -\frac{5}{2} + \frac{1}{2}\,{u_t}^2 - \frac{\alpha}{2}\,u\,u_{xx} + \frac{1}{2}\,u u_{xxxx} + \cos\,u + \frac{3}{2}\,\cos^2\,u,$$

$$T_1^2 = -\frac{1}{2} u u_{txxx} + \frac{1}{2} u_x u_{txx} - \frac{1}{2} u_{xx} u_{tx} + \frac{1}{2} u_{xxx} u_t + \frac{\alpha}{2} u u_{tx} - \frac{\alpha}{2} u_x u_t$$

and

$$\begin{aligned} T_2^1 &= -\frac{1}{2} u u_{tx} + \frac{1}{2} u_t u_x, \\ T_2^2 &= -\frac{5}{2} + \cos u + \frac{3}{2} \cos^2 u - \frac{\alpha}{2} {u_x}^2 + u_x u_{xxx} + \frac{1}{2} u u_{tt} - \frac{1}{2} {u_{xx}}^2. \end{aligned}$$

7.3.2 Conservation laws for the Boussinesq-double sinh-Gordon equation (7.2)

In this subsection, we give conservation laws of the Boussinesq-double sinh-Gordon equation which where obtain by using the Noether theorem and multiplier method.

Application of Noether theorem

Equation (7.2) has the Lagrangian given by

$$L = -\frac{1}{2}u_t^2 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_{xx}^2 - \sinh u - \frac{3}{4}\sinh(2u). \tag{7.23}$$

The application of Noether theorem gives the conserved vectors of (7.2) as

$$T_1^1 = \frac{1}{2}u_t^2 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_{xx}^2 - \cosh u - \frac{3}{4}\cosh(2u), \quad T_1^2 = -\alpha u_t u_x + u_t u_{xxx} - u_{xt} u_{xx}$$

and

$$T_2^1 = u_t u_x, \quad T_2^2 = -\frac{1}{2}u_t^2 - \frac{\alpha}{2}u_x^2 - \frac{1}{2}u_{xx}^2 + u_x u_{xxx} - \cosh u - \frac{3}{4}\cosh(2u).$$

Application of the multiplier method

Equation (7.2) has the following first-order multiplier:

$$\Lambda(t, x, u, u_t, u_x) = C_1 u_t + C_2 u_x. \tag{7.24}$$

The application of the multiplier method yields the following conserved quantities of (7.2):

$$T_1^1 = \frac{5}{2} + \frac{1}{2} {u_t}^2 - \frac{\alpha}{2} u u_{xx} + \frac{1}{2} u u_{xxxx} - \cosh u - \frac{3}{2} \cosh^2 u,$$

$$T_1^2 = -\frac{1}{2}uu_{txxx} + \frac{1}{2}u_x u_{txx} - \frac{1}{2}u_{xx}u_{tx} + \frac{1}{2}u_{xxx}u_t + \frac{\alpha}{2}u\,u_{tx} - \frac{\alpha}{2}u_x u_t$$

and

$$T_2^1 = -\frac{1}{2}uu_{tx} + \frac{1}{2}u_t u_x,$$
$$T_2^2 = \frac{5}{2} - \cosh u - \frac{3}{2}\cosh^2 u - \frac{\alpha}{2}{u_x}^2 + u_x u_{xxx} + \frac{1}{2}uu_{tt} - \frac{1}{2}{u_{xx}}^2.$$

7.3.3 Conservation laws for the Boussinesq-Liouville type I equation (7.3)

In this subsection, we use the Noether theorem and the multiplier method to obtain conservation laws of the Boussinesq-Liouville type I equation.

Application of Noether theorem

The second-order Lagrangian for equation (7.3) is given by

$$L = -\frac{1}{2}u_t^2 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_{xx}^2 - e^u - \frac{3}{8}e^{2u}. \tag{7.25}$$

The use of the theorem due to Noether with the Noether operator X_1, gives the conserved vector

$$T_1^1 = \frac{1}{2}u_t^2 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_{xx}^2 - e^u - \frac{3}{8}e^{2u}, \quad T_1^2 = -\alpha u_t u_x + u_t u_{xxx} - u_{xt}u_{xx}.$$

Using $X_2 = \partial_x$ and applying the Noether theorem, we obtain

$$T_2^1 = u_t u_x, \quad T_2^2 = -\frac{1}{2}u_t^2 - \frac{\alpha}{2}u_x^2 - \frac{1}{2}u_{xx}^2 + u_x u_{xxx} - e^u - \frac{3}{8}e^{2u}.$$

Application of the multiplier method

Equation (7.3) has the following first-order multiplier:

$$\Lambda(t, x, u, u_t, u_x) = C_1 u_t + C_2 u_x. \tag{7.26}$$

The application of the multiplier method yields the following conserved quantities for (7.3):

$$T_1^1 = \frac{11}{8} + \frac{1}{2}{u_t}^2 - \frac{\alpha}{2}u\,u_{xx} + \frac{1}{2}uu_{xxxx} - e^u - \frac{3}{8}e^{2u},$$

$$T_2^1 = \frac{\alpha}{2}\, u\, u_{tx} - \frac{\alpha}{2}\, u_x\, u_t - \frac{1}{2}\, u u_{txxx} + \frac{1}{2}\, u_x\, u_{txx} - \frac{1}{2}\, u_{xx}\, u_{tx} + \frac{1}{2}\, u_{xxx}\, u_t$$

and

$$T_2^1 = -\frac{1}{2}\, u u_{tx} + \frac{1}{2}\, u_t\, u_x,$$
$$T_2^2 = \frac{11}{8} - \frac{\alpha}{2}\, {u_x}^2 + u_x\, u_{xxx} + \frac{1}{2}\, u u_{tt} - \frac{1}{2}\, {u_{xx}}^2 - e^u - \frac{3}{8}\, e^{2u}.$$

7.3.4 Conservation laws for the Boussinesq-Liouville type II equation (7.4)

In this subsection, we construct conservation laws of the Boussinesq-Liouville type II equation by using the Noether theorem and the multiplier method.

Application of Noether theorem

The Lagrangian of equation (7.4) is given by

$$L = -\frac{1}{2}u_t^2 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_{xx}^2 + e^{-u} + \frac{3}{8}e^{-2u}. \tag{7.27}$$

Similarly by using the theorem due to Noether with $X_1 = \partial_t$, gives the conserved vector

$$T_1^1 = \frac{1}{2}u_t^2 + \frac{\alpha}{2}u_x^2 + \frac{1}{2}u_{xx}^2 + e^{-u} + \frac{3}{8}e^{-2u}, \quad T_1^2 = -\alpha u_t u_x + u_t u_{xxx} - u_{xt} u_{xx}$$

and using $X_2 = \partial_x$ together with the Noether theorem, we obtain

$$T_2^1 = u_t u_x, \quad T_2^2 = -\frac{1}{2}u_t^2 - \frac{\alpha}{2}u_x^2 - \frac{1}{2}u_{xx}^2 + u_x u_{xxx} + e^{-u} + \frac{3}{8}e^{-2u}.$$

Application of the multiplier method

The first-order multiplier for equation (7.4) is given by

$$\Lambda(t, x, u, u_t, u_x) = C_1 u_t + C_2 u_x. \tag{7.28}$$

Thus, the multiplier method yields the following local conserved quantities for (7.4):

$$T_1^1 \;=\; -\frac{11}{8} + \frac{3}{8}e^{-2u} + \frac{1}{2}\, {u_t}^2 - \frac{\alpha}{2}\, u u_{xx} + \frac{1}{2}\, u u_{xxxx} + e^{-u},$$

$$T_1^2 = \frac{\alpha}{2} u u_{tx} - \frac{1}{2} u u_{txxx} + \frac{1}{2} u_{xxx} u_t - \frac{\alpha}{2} u_x u_t + \frac{1}{2} u_x u_{txx} - \frac{1}{2} u_{xx} u_{tx}$$

and

$$T_2^1 = -\frac{1}{2} u u_{tx} + \frac{1}{2} u_t u_x,$$
$$T_2^2 = -\frac{11}{8} + \frac{3}{8} e^{-2u} - \frac{1}{2} {u_{xx}}^2 - \frac{\alpha}{2} {u_x}^2 + \frac{1}{2} u u_{tt} + e^{-u} + u_x u_{xxx}.$$

7.4 Concluding remarks

In this chapter we studied the four Boussinesq-type equations, namely the Boussinesq-double sine-Gordon equation, the Boussinesq-double sinh-Gordon equation, the Boussinesq-Liouville type I equation and the Boussinesq-Liouville type II. Lie point symmetries of these equations were obtained and the two translation symmetries were used to transform these equations into ODEs. Then the simplest equation method was used to obtain exact solutions. Futhermore, several conserved quantities for these equations were derived by employing two different techniques, namely the Noether theorem and multiplier method.

Chapter 8

Concluding remarks

In this thesis we first recalled some important definitions and results from Lie group theory and conservation laws, which were later used. In Chapter two, Lie group classification was performed on the generalized Klein-Gordon equation in (2+1) dimensions. The functional forms of (2.1) of the type linear, power, exponential and logarithmic were obtained. We retrieved two special equations, namely, the Liouville equation in (2+1) dimension and the (2+1)-dimensional generalized combined sinh-cosh-Gordon equation. In addition, the group-invariant solutions were derived for power and exponential functions. We have also illustrated that the (2+1)-dimensional Klein-Gordon equation is nonlinearly self-adjoint. In addition, conservation laws for the nonlinearly self-adjoint subclasses were derived by using the new Ibragimov's theorem.

In Chapter three we studied the generalized double sinh-Gordon equation (3.1) using the Lie symmetry analysis. Symmetry reductions based on the optimal systems of one-dimensional subalgebras of (3.1) and exact solutions with the help of simplest equation method and exponential-function method were obtained. Conservation laws for (3.1) were derived by employing four different methods, namely the direct method, Noether theorem, the new conservation theorem and multiplier method.

In Chapter four we investigated the generalized double combined sinh-cosh-Gordon equation from the point of view of Lie symmetry analysis. Similarity reductions and exact solutions with the aid of simplest equation method were obtained based on the optimal systems of one-dimensional subalgebras for the generalized double combined sinh-cosh-Gordon equation. Finally, local conserved vectors for the generalized double combined sinh-cosh-Gordon equation were derived by various methods.

In Chapter five Lie group analysis was utilized to study the (2+1)-dimensional nonlinear sinh-Gordon equation. We also obtained exact solutions for this equation by using the symmetry analysis along with the simplest equation method and (G'/G)-expansion method. Furthermore, several conserved quantities for equation (5.1) were derived by employing three different techniques; the direct method, the Noether theorem and the new conservation theorem.

In Chapter six Lie symmetry analysis was employed to study the (3+1)-dimensional nonlinear sinh-Gordon equation. Exact solutions of the underlying equation were obtained by using Lie symmetry analysis together with the simplest equation method and (G'/G)-expansion method. Conservation laws were also constructed by using the Noether theorem and the new conservation theorem.

In Chapter seven we studied the four Boussinesq-type equations, namely, the Boussinesq-double sine-Gordon equation, the Boussinesq-double sinh-Gordon equation, the Boussinesq-Liouville type I equation and the Boussinesq-Liouville type II. Lie point symmetries of these equations were obtained and the two translation symmetries were used to transform these equations into ODEs. Thereafter, the simplest equation method was used to obtain exact solutions. Conserved quantities for these equations were derived by employing two different techniques; namely, the Noether theorem and multiplier method.

In future, conservation laws of various nonlinear differential equations will be used to construct exact solutions.

Chapter 9

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