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The Controversial Millennium Problem – Proof that NP=P

Topic

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THE CONTROVERSIAL MILLENNIUM PROBLEM – PROOF THAT NP=P

ABSTRACT

The general binary linear programming (BLP) problem is known to be NP Complete. The lecture presents a new approach of transforming any BLP into a convex quadratic programming (CQP) problem. It is known that the CQPs can be solved by interior point algorithms in polynomial time (P). This implies that NP=P and settles one of the controversial millennium open problems.

KEYWORDS

NP – complete, binary linear programming, convex function, convex quadratic programming problem, interior point algorithm and polynomial time.

1. INTRODUCTION

Background

Binary linear programming (BLP) is *NP*-complete and up to now we have not been aware of any other polynomial algorithm for this problem other than the one proposed by Munapo in 2016 [9]. See for example Fortnow [4,5,6] for more on complexity. In this paper, we present a technique for transforming the BLP model into a convex quadratic programming (CQP) problem [3,8]. The optimal solution of the resultant convex QP is also the optimal solution of the original problem BLP. Interior point algorithms can solve CQPS in polynomial time [7]. This solves one of the most famous controversial millennium open problems, which is P=NP or not?

Millennium problems

There are seven Millennium Prize Problems that were set by the Clay Mathematics Institute (CMI) in 2000 [2]. Of these seven, only one has been solved and six are yet to be solved, as of 31 March 2019. The seven Millennium problems are:

(i) P versus NP,
(ii) Hodge conjecture,
(iii) Riemann hypothesis,
(iv) Yang-Mills existence and mass gap,
(v) Navier-Stokes existence and smoothness,
(vi) Birch and Swinnerton-Dyer conjecture and
(vii) Poincaré conjecture.

Each problem is worth $1 000 000 and CMI is ready to give the prize money for anyone who can solve these difficult problems. Of the seven millennium problems, only one has been solved. The last one (Pointcare conjecture) was solved in 2003 by a Russian mathematician Grigori Perelman. He declined the award and the prize money that were officially given to him in 2010.

2. THE BLP MODEL

Let any BLP model be represented by

Maximise CX^T,

Such that: $AX^T \le B^T,\ X^T \le I^T,\ X^T \ge 0,$

Where $I = (1 \quad 1 \quad \ldots \quad 1),\ A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix},$ (1)

$B = (b_1 \quad b_2 \quad \ldots \quad b_m).,\ C = (c_1, c_2, \ldots, c_n),\ X = (x_1 \quad x_2 \quad \ldots \quad x_n).$

Any minimisation BLP can be converted into maximisation form and *vice versa*. There are several strategies for solving mixed 0-1 integer problems that are presented in Adams and Sherali [1].

3. CONVEX QUADRATIC PROGRAMMING MODEL

Let a quadratic programming problem be represented by (2).

$$
\left.
\begin{aligned}
&\text{Minimise } f(X) = CX^T + \tfrac{1}{2} XQX^T, \\
&\text{Such that: } \quad AX^T \le B^T, \\
&\qquad X^T \le I^T, \\
&\qquad X^T \ge 0. \\
&\text{Where } \quad Q = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \cdots & \cdots & \cdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix}. \\
&\text{We assume that:}
\end{aligned}
\right\} \qquad (2)
$$

(i) matrix Q is symmetric and positive definite,
(ii) function $f(X)$ is strictly convex,
(iii) since constraints are linear, the solution space is convex,
(iv) any maximisation quadratic problem can be changed into a minimisation and *vice versa*.

When the function $f(X)$ is strictly convex for all points in the convex region, then the quadratic problem has a unique local minimum that is also the global minimum [3,8,11].

4. TRANSFORMING BLP INTO A CONVEX/CONCAVE QUADRATIC PROGRAMMING PROBLEM

Our problem is to transform problem (1) into (2), and once that is done, then (2), which can be solved in polynomial time, implying P=NP. Interior point algorithms can solve the convex/concave QP problem in polynomial time.

4.1 Rules with binary variables

Binary variables have certain special features that we can capitalise on when solving. These features are given as rules 1 and 2.

4.1.1 Rule 1

Given any binary variable x_j, which in this case is supposed to be integer, then its slack s_j is also binary in the optimal solution.

Proof

$$x_j + s_j = 1. \tag{3}$$

Case 1: When $x_j = 1$ then $s_j = 0.$

Case 2: When $x_j = 0$ then $s_j = 1.$

4.1.2 Rule 2

For any binary variable x_j and slack variable s_j, the following must hold at optimality for BLPs.

$$x_j^2 + s_j^2 = 1. \tag{4}$$

The proof is similar to the one in rule 1. Note that it is only a binary variable x_j that can satisfy (4).

Case 1: When $x_j = 1$ then $s_j = 0.$ This case satisfies (4).

Case 2: When $x_j = 0$ then $s_j = 1.$ Equation (4) is also satisfied.

Case 3: When x_j is not integer.

This implies $0 < x_j < 1$ and $0 < s_j < 1$.

All fractions between 0 and 1 become smaller when they are squared.

i.e. $x_j^2 < x_j$ and $s_j^2 < s_j$ for example $(0.3)^2 = 0.09 < 0.3$.

Suppose $x_j + s_j = 1$, $0 < x_j < 1$ and $0 < s_j < 1$.

If $(x_j^2 < x_j) + (s_j^2 < s_j)$ then

$$(x_j^2 + s_j^2) < (x_j + s_j). \tag{5}$$

Also

$$(x_j^2 + s_j^2) \neq 1. \tag{6}$$

In other words, there are no non-integer values that can satisfy (4). Even though there are some non-binary values that can satisfy (3), such values as cannot satisfy (4). For example, if $x_j = 0.9$, then automatically $s_j = 0.1$ and this can satisfy (3). The same values cannot satisfy (4), i.e. $0.9^2 + 0.1^2 = 0.82 \neq 1$. The binary variable and slack variable relationship given in (4) forms the pillar or backbone of this lecturer.

4.2 Forcing variables to assume binary variables

The main weakness of the objective function given in (1) is that it does not force variables to assume binary values. In this paper, we alleviate this challenge by adding a nonlinear extension to the objective function as given in (7).

$$\text{Maximise } C\overline{X}^T + \ell\overline{X}(\overline{X}^T) \tag{7}$$

Where $\overline{X} = (x_1 \quad x_2 \quad \ldots \quad x_n \, s_1 \quad s_2 \quad \ldots \quad s_n)$. and ℓ is a very large constant. The constant ℓ is very large in terms of its size compared to any of the coefficients in the objective function. This large value can be approximated as (8).

$$\ell = 1000(|c_1| + |c_2| + ... + |c_n|) \quad (8)$$

Proof

$$\ell\overline{X}(\overline{X}^T) = \ell(x_1^2 + x_2^2 + ... + x_n^2 + s_1^2 + s_2^2 + ... + s_n^{2)}), \quad (9)$$

$$\ell\overline{X}(\overline{X}^T) = \ell((x_1^2 + s_1^2) + (x_2^2 + s_2^2) + ... + (x_n^2 + s_n^2)). \quad (10)$$

Since from rule 2, $x_j^2 + s_j^2 = 1$, then $C\overline{X}^T + \ell\overline{X}(\overline{X}^T)$ is maximised when

$$(x_1^2 + s_1^2 = 1), (x_2^2 + s_2^2 = 1), ..., (x_n^2 + s_n^2 = 1). \quad (11)$$

In other words, $C\overline{X}^T + \ell\overline{X}(\overline{X}^T)$ is maximised when variable x_j and slack variable s_j are integers. In this lecture, we call the nonlinear extension $\ell\overline{X}(\overline{X}^T)$, which is called an *enforcer*. An enforcer is a function, a set of constraint(s) or combination of both added to a problem to force an optimal solution with desired features such integrality.

4.3 Convexity of $C\overline{X}^T + \ell\overline{X}(\overline{X}^T)$

A function $f(\overline{X}) = f(x_1, x_2, ..., x_n, s_1, s_2, ..., s_n)$ is convex if and only if it has second-order partial derivatives for each point $\overline{X} = (x_1, x_2, ..., x_n, s_1, s_2, ..., s_n) \in S$ and for each $\overline{X}' \in S$ all principal minors of the Hessian matrix are none negative.

Proof

In this case, the function $f(\overline{X})$ is given as (12).

$$\begin{aligned} f(\overline{X}) &= f(x_1, x_2, ..., x_n, s_1, s_2, ..., s_n) \\ &= c_1x_1 + c_2x_2 + ... + c_nx_n + \ell(x_1^2 + x_2^2 + ... + x_n^2 + s_1^2 + s_2^2 + ... + s_n^3). \end{aligned} \quad (12)$$

This has continuous second order partial derivatives and the 2*n* by 2*n* Hessian matrix is given as (13).

$$H(x_1, x_2, ..., x_n, s_1, s_2, ..., s_n) = \begin{bmatrix} 2\ell & 0 & ... & 0 \\ 0 & 2\ell & ... & 0 \\ ... & & & ... \\ 0 & 0 & ... & 2\ell \end{bmatrix}. \quad (13)$$

Since all principal minors of $H(x_1, x_2, ..., x_n, s_1, s_2, ..., s_n)$ are nonnegative, then $f(x_1, x_2, ..., x_n, s_1, s_2, ..., s_n)$ is convex. See Winston [12] for more on convex functions.

4.4 Convex quadratic programming form

The function $C\overline{X}^T + \ell\overline{X}(\overline{X}^T)$ can be expressed in the convex quadratic programming form (14).

$$f(\overline{X}) = C\overline{X}^T + \tfrac{1}{2}(2\ell\overline{X}\overline{X}^T),$$

$$\Rightarrow f(\overline{X}) = C\overline{X} + \tfrac{1}{2}\overline{X}\tilde{Q}\overline{X}^T. \quad (14)$$

Where matrix $\tilde{Q}$ is of dimension 2*n* by 2*n*, symmetric and positive definite, as given in (15).

$$\tilde{Q} = \begin{bmatrix} 2\ell & 0 & ... & 0 \\ 0 & 2\ell & ... & 0 \\ ... & & & ... \\ 0 & 0 & ... & 2\ell \end{bmatrix}. \quad (15)$$

Therefore, matrix $\tilde{Q}$ is symmetric and positive definite. Note that $\overline{X}\tilde{Q}\overline{X}^T \geq 0, \forall \overline{X}^T \geq 0.$

4.5 Complexity of convex quadratic programming

The main reason for converting a BLP into a convex quadratic programming model is to take advantage of the availability of interior point algorithms that can solve convex QPs in polynomial time [7]. If any BLP can be converted into a convex quadratic problem, then any BLP can be solved in polynomial time.

4.6 Proof of optimality

The proof can be easily shown by reducing the convex quadratic objective function to the original linear form given in (1). The proposed objective function of the convex QP is reduced as follows:

$$\text{Maximise } c_1x_1 + c_2x_2 + \ldots + c_nx_n + \ell(x_1^2 + x_2^2 + \ldots + x_n^2 + s_1^2 + s_2^2 + \ldots + s_n^3),$$

$$= \text{Maximise } c_1x_1 + c_2x_2 + \ldots + c_nx_n + \ell((x_1^2 + s_1^2) + (x_2^2 + s_2^2) + \ldots + (x_n^2 + s_n^2)).$$

Since $x_j^2 + s_j^2 = 1, \forall j$ then,

$$\text{Maximise } c_1x_1 + c_2x_2 + \ldots + c_nx_n + \ell((1) + (1) + \ldots + (1), \tag{16}$$

$$=\text{Maximise } c_1x_1 + c_2x_2 + \ldots + c_nx_n + n\ell. \tag{17}$$

In other words, $\ell(x_1^2 + x_2^2 + \ldots + x_n^2 + s_1^2 + s_2^2 + \ldots + s_n^3)$ is a constant and the objective function is the same as just, maximise $c_1x_1 + c_2x_2 + \ldots + c_nx_n$, where x_j is binary for $j = 1, 2, \ldots, n$, which is the original form given in (1).

4.6 Infeasible binary integer solution space

In this case, the solution of the convex OP will not be integer. The objective,

Maximise $c_1x_1 + c_2x_2 + \ldots + c_nx_n + \ell(x_1^2 + x_2^2 + \ldots + x_n^2 + s_1^2 + s_2^2 + \ldots + s_n^3),$ forces variables to be binary or integer. If an integer point does not exist in the solution space, then the large constant ℓ in the objective forces variables to assume values whose sums of squares are near 1 and not necessarily 1. In other words, the variables will assume values x'_j and s'_j such that

$$(x'_j)^2 + (s'_j)^2 < 1, \tag{18}$$

i.e. $$(x'_j)^2 + (s'_j)^2 \approx 1. \tag{19}$$

4.7 Mixed BLP models

In some BLP problems that occur in real life, a set of the variables may not be restricted to integer values. In this case, the enforcer $\ell\bar{X}(\bar{X}^T)$ is composed of only those variables that are supposed to be binary or integer.

4.7 Interior point algorithm for convex QP

Any maximisation BLP problem can be converted into a minimisation BLP and *vice versa*. This can be done by the substitution given in (20).

$$x_j = 1 - \bar{x}_j. \tag{20}$$

Where $\bar{x}_j$ is also a binary variable.

Suppose the primal-dual pair of the convex QP is given by (21) and (22).

Primal:

$$\left.\begin{array}{l} \text{Minimise } CX^T + \tfrac{1}{2} XQX^T, \\ \text{Such that:} \\ AX^T = B^T, \\ X^T \geq 0,. \end{array}\right\} \tag{21}$$

Dual:

$$\left.\begin{array}{l}\text{Maximise } B^T Y - \tfrac{1}{2} X Q X^T, \\ \\ \text{Such that:} \\ \\ A^T Y + \mu - Q X^T = C^T.\end{array}\right\} \quad (22)$$

Where Y is free, $\mu \geq 0$ and μ is a diagonal matrix.

The first-order optimality conditions for (21) and (22) are given by (23)

$$\left.\begin{array}{c} A X^T = B^T, \\ \\ A^T Y + \mu - Q X^T = C^T, \\ \\ X^T \mu e = 0, \\ \\ X^T \geq 0, \\ \\ \mu \geq 0. \end{array}\right\} \quad (23)$$

Where e is a vector of ones. The primal-dual central path method can be used to solve the convex QP. Detailed information on this interior point algorithm and other variants can be obtained in Gondzio [7].

5. BLP AND CONVEX QP RELATIONSHIP

Maximise $c_1x_1 + c_2x_2 + ... + c_nx_n$,

Subject to

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1,$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2}x_n \leq b_2,$$

$$\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m,$$

$$x_j \leq 1, j = 1,2,...,n,$$

x_j is integer $\forall_j$.

NP-Complete form

$$\text{Maximise } \begin{array}{l} c_1x_1 + c_2x_2 + \ldots + c_nx_n + \\ \ell(x_1^2 + x_2^2 + \ldots + x_n^2 + s_1^2 + s_2^2 + \ldots + s_n^3), \end{array}$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1,$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_2x_n \le b_2,$$

$$\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m,$$

$$x_j + s_j = 1, j = 1,2,\ldots,n\ ,$$

$$\ell = 1000(|c_1| + |c_2| + \ldots + |c_n|),$$

$$x_j \ge 0.$$

P form

From the two versions of the same problem

$$NP = P \tag{24}$$

6. NUMERICAL ILLUSTRATION

The following numerical illustration shows how a BLP problem is transformed into convex quadratic programming model and then solved.

6.1 Pure binary linear programming

$$\left.\begin{aligned}
&\text{Maximise } 3x_1 + 14x_2 + 3x_3 + 8x_4 + 4x_5,\\
&\text{Such that: } 10x_1 + 12x_2 + 4x_3 + 6x_4 + 13x_5 \le 20,\\
&17x_1 - 22x_2 + 35x_3 + 8x_4 + 18x_5 \le 25,\\
&-10x_1 + 8x_2 + 23x_3 + 11x_4 - 6x_5 \ge 18.
\end{aligned}\right\} \quad (25)$$

Where $x_1, x_2, x_3, x_4, x_5 \ge 0$ are binary variables.

Transforming into a convex quadratic programming problem becomes (26).

$$\left.\begin{aligned}
&\text{Maximise } 3x_1 + 14x_2 + 3x_3 + 8x_4 + 4x_5 + 32000(x_1^2 + x_2^2 +\\
&\quad x_3^2 + x_4^2 + x_5^1 + s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2),\\
&\text{Such that: } 10x_1 + 12x_2 + 4x_3 + 6x_4 + 13x_5 \le 20,\\
&17x_1 - 22x_2 + 35x_3 + 8x_4 + 18x_5 \le 25,\\
&-10x_1 + 8x_2 + 23x_3 + 11x_4 - 6x_5 \ge 18,\\
&x_1 + s_1 = 1,\\
&x_2 + s_2 = 1,\\
&x_3 + s_3 = 1,\\
&x_4 + s_4 = 1,\\
&x_5 + s_5 = 1.
\end{aligned}\right\} \quad (26)$$

,

Where $s_1, s_2, s_3, s_4, s_5 \geq 0$.

The solution to the convex quadratic problem is given in (27).

$$x_2 = x_4 = s_1 = s_3 = s_5 = 1 \text{ and } x_1 = x_3 = x_5 = s_2 = s_4 = 0. \tag{27}$$

6.2 Mixed binary linear programming problem

In the case of a mixed binary linear programming problem, only the binary integer variables occupy the enforcer. In other words, if only the first *r* variables $(x_1 \quad x_2 \quad \ldots \quad x_r)$ are integer, then we use (28).

$$\text{Maximise } c_1x_1 + c_2x_2 + \ldots + c_nx_n + \ell((x_1^2 + s_1^2) + (x_2^2 + s_2^2) + \ldots + (x_r^2 + s_r^2)) \tag{28}$$

Suppose in 5.1, the variables x_1 and x_2 are not restricted to integer, but both variables are less than 1. The transformation becomes as shown in (29).

$$\left.\begin{aligned}
&\text{Maximise} \quad 3x_1 + 14x_2 + 3x_3 + 8x_4 + 4x_5 + 32000(x_3^2 + x_4^2 + x_5^2 + s_3^2 + s_4^2 + s_5^2) \\
&\text{Such that:} \quad 10x_1 + 12x_2 + 4x_3 + 6x_4 + 13x_5 \leq 20, \\
&17x_1 - 22x_2 + 35x_3 + 8x_4 + 18x_5 \leq 25, \\
&-10x_1 + 8x_2 + 23x_3 + 11x_4 - 6x_5 \geq 18, \\
&x_3 + s_3 = 1, \; x_4 + s_4 = 1, \; x_5 + s_5 = 1, \\
&x_1 \leq 1, \; x_2 \leq 1.
\end{aligned}\right\} \tag{29}$$

The solution to the convex quadratic problem is given in (30).

$$x_2 = 0.833, x_3 = x_4 = s_5 = 1 \text{ and } x_1 = x_5 = s_3 = s_4 = 0. \tag{30}$$

7. FROM MIXED INTEGER PROBLEM TO BLP

The problems that occur in real life do not have binary variables only. These practical problems occur as general mixed integer problems (MIPs), where variables assume integer values greater than 1. There are methods that can be used to solve these problems, but we are not aware of any method that can solve these mixed integer problems in polynomial time up to now. The obvious strategy is to expand the general mixed integer variable into binary ones.

7.1 Converting MIP into BLP

Any MIP variable (x_j^g) can be expanded into binary variables as given in (31).

$$x_j^g = x_0^j + 2^1 x_1^j + 2^2 x_2^j + \ldots + 2^k x_k^j. \tag{31}$$

Where x_i^j is a binary variable for $i = 0,1,2,\ldots,k.$ This procedure is explained in Owen and Mehrotra [1,8].

7.2 Numerical illustration

$$\left.\begin{aligned}
&\text{Convert the following MIP into a BLP.}\\
&\text{Maximise } 6x_1 + 10x_2 + 14x_3 + 5x_4,\\
&\text{Such that } 8x_1 + 12x_2 + 7x_3 + 15x_4 \leq 52.\\
&\text{Where } x_1, x_2, x_3, x_4 \geq 0 \text{ are integers.}
\end{aligned}\right\} \tag{32}$$

The following substitutions given in (33) change the problem into a BLP.

$$
\left.
\begin{aligned}
x_1 &= x_0^1 + 2x_1^1 + 4x_2^1, \\
x_2 &= x_0^2 + 2x_1^2 + 4x_2^2, \\
x_3 &= x_0^3 + 2x_1^3 + 4x_2^3. \\
x_4 &= x_0^4 + 2x_1^4.
\end{aligned}
\right\} \tag{33}
$$

Where x_i^j is a binary variable for $i = 0,1,2$ and $j = 1,2,3.4.$

8. CONCLUSIONS

The general BLP problem has been given so much attention by researchers all over the world for over half a century without a breakthrough. A difficult category of BLP models includes the traveling salesman, generalised assignment, quadratic assignment and set covering problems. We presented a technique to solve BLP problems by first transforming them into convex QPs and then applying interior point algorithms to solve in polynomial time. We also showed that the proposed technique works for both pure and mixed BLPs and also for the general linear integer model where variables are expanded into BLPs. We hope the proposed approach will give more clues to researchers in the hunt for efficient solutions to the general difficult integer programming problem.

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