

Collision properties of overtaking supersolitons with small amplitudes

C. P. Olivier, F. Verheest, and W. A. Hereman

Citation: *Physics of Plasmas* **25**, 032309 (2018); doi: 10.1063/1.5027448

View online: <https://doi.org/10.1063/1.5027448>

View Table of Contents: <http://aip.scitation.org/toc/php/25/3>

Published by the *American Institute of Physics*

Articles you may be interested in

[The effects of finite mass, adiabaticity, and isothermality in nonlinear plasma wave studies](#)
Physics of Plasmas **25**, 032303 (2018); 10.1063/1.5019438

[Existence domain of electrostatic solitary waves in the lunar wake](#)
Physics of Plasmas **25**, 022302 (2018); 10.1063/1.5017638

[Electron-acoustic solitary waves in the Earth's inner magnetosphere](#)
Physics of Plasmas **25**, 022905 (2018); 10.1063/1.5007907

[Generation of ion acoustic solitary waves through wave breaking in superthermal plasmas](#)
Physics of Plasmas **24**, 102127 (2017); 10.1063/1.4991467

[New method for rekindling the nonlinear solitary waves in Maxwellian complex space plasma](#)
Physics of Plasmas **25**, 043703 (2018); 10.1063/1.5011317

[Ion acoustic solitary structures in a magnetized nonthermal dusty plasma](#)
Physics of Plasmas **25**, 033704 (2018); 10.1063/1.5021127



**COMPLETELY
REDESIGNED!**

**PHYSICS
TODAY**

Physics Today Buyer's Guide
Search with a purpose.

Collision properties of overtaking supersolitons with small amplitudes

C. P. Olivier,^{1,a)} F. Verheest,^{2,3,b)} and W. A. Hereman^{4,c)}

¹Centre for Space Research, North-West University, Potchefstroom 2520, South Africa

²Sterrenkundig Observatorium, Universiteit Gent, Krijgslaan 281, B-9000 Gent, Belgium

³School of Chemistry and Physics, University of KwaZulu-Natal, Durban 4000, South Africa

⁴Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, Colorado 80401-1887, USA

(Received 2 March 2018; accepted 13 March 2018; published online 28 March 2018)

The collision properties of overtaking small-amplitude supersolitons are investigated for the fluid model of a plasma consisting of cold ions and two-temperature Boltzmann electrons. A reductive perturbation analysis is performed for compositional parameters near the supercritical composition. A generalized Korteweg-de Vries equation with a quartic nonlinearity is derived, referred to as the modified Gardner equation. Criteria for the existence of small-amplitude supersolitons are derived. The modified Gardner equation is shown to be not completely integrable, implying that supersoliton collisions are inelastic, as confirmed by numerical simulations. These simulations also show that supersolitons may reduce to regular solitons as a result of overtaking collisions. *Published by AIP Publishing.*
<https://doi.org/10.1063/1.5027448>

I. INTRODUCTION

Supersolitons are deformed solitary waves that are distinguishable through their three local minima and three local maxima in the electric field. Since the first reports on supersolitons,^{1–3} an increasing number of plasma models that support supersolitons has been identified.^{4–9} Many of these models describe magnetospheric plasmas.

Regardless, very few actual satellite observations of possible supersoliton profiles have been reported.^{5,10} The limitations of spacecraft data mean that the time evolution of these structures cannot be traced. It is therefore nearly impossible to distinguish between supersolitons and regular soliton collisions.

The observed supersoliton-like structures^{5,10} are typically sandwiched between regular solitons, or more complicated electric field structures. This is not entirely unexpected, as solitons in space plasmas are usually observed in clusters.^{11–14} These observations suggest that supersolitons in space plasmas would frequently collide with other solitons. Therefore, it is important to understand the collision properties of supersolitons.

A fluid simulation was recently performed by Kakad *et al.*¹⁵ in order to investigate the properties of supersolitons. They simulated the formation of a supersoliton from a Gaussian initial density disturbance. The generated supersolitons are therefore stable and provide insight into the possible formation of supersolitons. However, the collision properties of the resulting supersolitons were not considered.

To date, theoretical studies have solely relied on pseudo-potential analysis due to Sagdeev.¹⁶ This approach is useful to obtain supersoliton solutions from which exact information about their amplitudes, velocities, and parametric regions of

existence can be deduced. Unfortunately, the study of collision properties falls outside the scope of Sagdeev analysis.

To study collision properties, we will apply the reductive perturbation analysis of Washimi and Taniuti.¹⁷ Previously, it was suggested that reductive perturbation analysis cannot be used to obtain small-amplitude supersolitons.^{5,18} But at that time, the existence of supercritical plasma compositions¹⁹ had not been reported yet. More recently, supercritical plasma compositions have been shown to be related to small-amplitude supersolitons.²⁰

In this paper, we show how this relationship can be used to study small-amplitude supersolitons by means of reductive perturbation analysis. This requires an extension of the earlier reductive perturbation analysis¹⁹ for a fluid plasma model consisting of cold ions and two-temperature Boltzmann electrons. The analysis leads to a generalized Korteweg-de Vries equation that admits supersoliton solutions. That equation is a higher order variant of the standard Gardner equation. Since we have not come across this equation previously, we refer to it as the modified Gardner (mG) equation.

The solutions obtained from this study agree exactly with those of an earlier small-amplitude study based on Sagdeev potential analysis.²⁰ The main advantage of the reductive perturbation analysis is that one may use the resulting evolution equation to analyze the collision properties of the supersolitons in the small amplitude regime. This is done in two ways. First, we show that the mG equation is not completely integrable. As a result, it follows that supersoliton collisions are inelastic. Second, we use the mG equation to simulate the collision between solitons and overtaking supersolitons. These simulations suggest that such collisions may reduce the supersoliton to a regular soliton with smaller amplitude.

It should be noted that our study is limited to very small regions in parameter space, very small amplitudes, and velocities that only marginally exceed the acoustic speed.²⁰ Indeed, a comprehensive study of supersoliton collisions can only be undertaken through full fluid simulations. However, our

^{a)}Electronic mail: carel.olivier@nwu.ac.za

^{b)}Electronic mail: frank.verheest@ugent.be

^{c)}Electronic mail: whereman@mines.edu

results show that the collision properties of small-amplitude supersolitons are very different from those of regular solitons.

The paper is organized as follows: In Sec. II, we present the fluid model. In Sec. III, we apply reductive perturbation analysis to derive the mG equation. We also establish the necessary conditions for the existence of supersoliton solutions. In Sec. IV, we normalize the mG equation and list its conservation laws. Moreover, we discuss why the equation is not completely integrable. Consequently, one should not expect collisions of solitons and supersolitons to be elastic. In Sec. V, we use the mG equation to simulate the collision of a supersoliton that overtakes a regular soliton. Some conclusions are drawn in Sec. VI together with an outlook on the future work.

II. FLUID MODEL

We consider a plasma consisting of cold fluid ions and a two-temperature Boltzmann electron species. The normalized fluid equations are given by¹⁹

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial x} = 0, \quad (2)$$

$$\frac{\partial^2 \phi}{\partial x^2} + n - f \exp(\alpha_c \phi) - (1-f) \exp(\alpha_h \phi) = 0, \quad (3)$$

where n denotes the ion number density normalized with respect to the equilibrium ion density N_i , and u is the fluid velocity normalized with respect to the ion-acoustic speed $c_{ia} = \sqrt{K_B T_{\text{eff}}/m_i}$ with ion mass m_i . Here, K_B denotes the Boltzmann constant and $T_{\text{eff}} = T_c[f + (1-f)\sigma]$ denotes the effective temperature with electron temperature ratio $\sigma = T_c/T_h$ for cool (hot, resp.) electron temperature T_c (T_h , resp.). The cool electron density f is normalized with respect to N_i . In addition, ϕ denotes the electrostatic potential normalized with respect to $K_B T_{\text{eff}}/e$ where e is the electron charge, while

$$\alpha_c = \frac{1}{f + (1-f)\sigma}, \quad (4)$$

and

$$\alpha_h = \frac{\sigma}{f + (1-f)\sigma}. \quad (5)$$

Finally, length x and time t are normalized with respect to the Debye length $\lambda_D = \sqrt{\epsilon_0 k T_{\text{eff}}/(N_i e^2)}$ and the reciprocal of the plasma frequency, $\omega_{pi}^{-1} = \sqrt{\epsilon_0 m_i/N_i e^2}$, respectively.

III. REDUCTIVE PERTURBATION ANALYSIS

In order to retain fourth-order nonlinear effects, we follow Ref. 19 and introduce a stretched coordinate system:

$$\xi = \varepsilon^{3/2}(x - t), \quad \tau = \varepsilon^{9/2}t. \quad (6)$$

In addition, we expand the ion number density and velocity, and the electrostatic potential as follows:

$$\begin{cases} n = 1 + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \varepsilon^4 n_4 + \dots, \\ u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + \dots, \\ \phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \varepsilon^4 \phi_4 + \dots. \end{cases} \quad (7)$$

Since we are interested in solitons and supersolitons, we impose the following boundary conditions:

$$n \rightarrow 1, \quad u \rightarrow 0, \quad \phi \rightarrow 0 \quad \text{when } |\xi| \rightarrow \infty. \quad (8)$$

By substituting the expressions (6) and (7) into the fluid equations (1)–(3), one obtains differential equations at different orders of ε . For brevity, we do not present these long expressions.

We start with the continuity equation (1). By substituting the expansions (6) and (7) into the continuity equation and collecting terms up to $\varepsilon^{11/2}$, one obtains the following equations:

$$n_{1\xi} = u_{1\xi}, \quad (9)$$

$$n_{2\xi} = u_{2\xi} + (n_1 u_1)_{\xi}, \quad (10)$$

$$n_{3\xi} = u_{3\xi} + (n_1 u_2 + n_2 u_1)_{\xi}, \quad (11)$$

$$n_{4\xi} = n_{1\tau} + u_{4\xi} + (n_1 u_3 + n_2 u_2 + n_3 u_1)_{\xi}. \quad (12)$$

The subscripts ξ and τ are used to denote partial derivatives $\partial/\partial\xi$ and $\partial/\partial\tau$, respectively. In addition, higher order partial derivatives are denoted with multiple subscripts throughout the paper. For example, we use $\phi_{1\xi\xi}$ to denote $\partial^2 \phi_1/\partial\xi^2$.

The first three equations (9)–(11) can be simplified by means of a simple integration. By taking the boundary conditions (8) into account, it follows that:

$$n_1 = u_1, \quad (13)$$

$$n_2 = u_2 + n_1 u_1, \quad (14)$$

and

$$n_3 = u_3 + n_1 u_2 + n_2 u_1. \quad (15)$$

A similar treatment of the momentum equation (2) produces the following set of equations:

$$u_1 = \phi_1, \quad (16)$$

$$u_2 = \phi_2 + \frac{1}{2} u_1^2, \quad (17)$$

$$u_3 = \phi_3 + u_1 u_2, \quad (18)$$

and

$$u_{4\xi} = u_{1\tau} + \phi_{4\xi} + u_1 u_{3\xi} + u_2 u_{2\xi} + u_3 u_{1\xi}. \quad (19)$$

The set of equations (16)–(19) can be combined to eliminate the u dependence from the set of equations (12)–(15). It follows that:

$$n_1 = \phi_1, \quad (20)$$

$$n_2 = \phi_2 + \frac{3}{2} \phi_1^2, \quad (21)$$

$$n_3 = \phi_3 + 3\phi_1 \phi_2 + \frac{5}{2} \phi_1^3, \quad (22)$$

and

$$n_{4\xi} = \phi_{4\xi} + 2\phi_{1\tau} + \left(2\phi_1\phi_3 + \frac{1}{2}\phi_2^2 + \frac{3}{2}\phi_1^2\phi_2 + \frac{5}{8}\phi_1^4 \right)_\xi. \quad (23)$$

We now turn to Poisson’s equation (3). By applying the expansions (6) and (7), using a Taylor series to expand the exponential functions, and retaining terms up to order ε^4 , one obtains the following equation:

$$\begin{aligned} &\varepsilon^4 \phi_{1\xi\xi\xi} + \varepsilon n_1 + \varepsilon^2 n_2 + \varepsilon^3 n_3 + \varepsilon^4 n_4 - \varepsilon A_1 \phi_1 - \varepsilon^2 A_1 \phi_2 \\ &- \varepsilon^3 A_1 \phi_3 - \varepsilon^4 A_1 \phi_4 - \frac{A_2}{2} \varepsilon^2 \phi_1^2 - A_2 \varepsilon^3 \phi_1 \phi_2 - A_2 \varepsilon^4 \phi_1 \phi_3 \\ &- \frac{A_2}{2} \varepsilon^4 \phi_2^2 - \frac{A_3}{6} \varepsilon^3 \phi_1^3 - \frac{A_3}{2} \varepsilon^4 \phi_1^2 \phi_2 - \frac{A_4}{24} \varepsilon^4 \phi_1^4 = 0, \end{aligned} \quad (24)$$

where

$$A_j = f\alpha_c^j + (1-f)\alpha_h^j. \quad (25)$$

Equations (20)–(22) must be substituted into (24). To use (23), we differentiate (24) with respect to ξ . Since $A_1 = 1$ for any choice of f and σ , Eq. (24) becomes

$$\begin{aligned} &\varepsilon^4 \phi_{1\xi\xi\xi\xi} + 2\varepsilon^4 \phi_{1\tau} + \left[\frac{3-A_2}{2} \varepsilon^2 \phi_1^2 + (3-A_2)\varepsilon^3 \phi_1 \phi_2 \right. \\ &+ \frac{15-A_3}{6} \varepsilon^3 \phi_1^3 + (3-A_2)\varepsilon^4 \phi_1 \phi_3 + \frac{15-A_3}{2} \varepsilon^4 \phi_1^2 \phi_2 \\ &\left. + \frac{3-A_2}{2} \varepsilon^4 \phi_2^2 + \frac{105-A_4}{24} \varepsilon^4 \phi_1^4 \right]_\xi = 0. \end{aligned} \quad (26)$$

For the supercritical plasma composition $f = \frac{1}{6}(3 - \sqrt{6})$ and $\sigma = 5 - 2\sqrt{6}$, one has $A_2 = 3$ and $A_3 = 15$, so that the terms in orders ε^2 and ε^3 in (26) vanish. Here, we consider plasma compositions near the supercritical composition. To do so, we look for compositions that satisfy the following criteria:

$$A_2 = 3 - \varepsilon^2 B_2, \quad A_3 = 15 - \varepsilon B_3. \quad (27)$$

We thus require that A_2 is close to 3 up to order ε^2 and that A_3 only differs from 15 by a quantity of order ε . Obviously, B_2 and B_3 must both be of order 1.

If we substitute (27) into (26) and retain terms of order ε^4 , we obtain the following equation:

$$\phi_{1\tau} + \frac{1}{2}\phi_{1\xi\xi\xi} + \frac{B_2}{2}\phi_1\phi_{1\xi} + \frac{B_3}{4}\phi_1^2\phi_{1\xi} + \frac{105-A_4}{12}\phi_1^3\phi_{1\xi} = 0. \quad (28)$$

For further analysis of (28), we consider the lowest order approximation of the electrostatic potential

$$\Phi = \varepsilon\phi_1. \quad (29)$$

In addition, we introduce the following changes of coordinates:

$$t = \varepsilon^{-9/2}\tau, \quad \eta = \varepsilon^{-3/2}\xi = x - t. \quad (30)$$

Then (28) becomes

$$\Phi_t + \frac{1}{2}\Phi_{\eta\eta\eta} + a\Phi\Phi_\eta + b\Phi^2\Phi_\eta + c\Phi^3\Phi_\eta = 0, \quad (31)$$

where

$$a = \frac{3-A_2}{2}, \quad b = \frac{15-A_3}{4}, \quad c = \frac{105-A_4}{12}. \quad (32)$$

To the best of our knowledge, (31) has not been reported before in the literature. We will refer to it as the modified Gardner (mG) equation since it is a quartic version of the standard Gardner equation where $c = 0$.

To find solitary wave solutions, we introduce a moving frame

$$\zeta = \eta - vt \quad (33)$$

and integrate the resulting ordinary differential equation twice to obtain the energy-like equation

$$\frac{1}{2}\frac{\partial}{\partial\zeta}(\Phi^2) + V(\Phi) = 0, \quad (34)$$

where

$$V(\Phi) = -v\Phi^2 + \frac{a}{3}\Phi^3 + \frac{b}{6}\Phi^4 + \frac{c}{10}\Phi^5. \quad (35)$$

Note that the above Sagdeev potential $V(\Phi)$ agrees with the one obtained in a small-amplitude study²⁰ based on a Taylor series expansion of the Sagdeev potential. We briefly summarize the main results from that paper:

- (1) For the model under consideration, a supercritical plasma composition exists for $\sigma = 5 - 2\sqrt{6}$ and $f = (3 - \sqrt{6})/6$, yielding $A_2 = 3$ and $A_3 = 15$. Using (32), it follows that $a = b = 0$ in (31).
- (2) For supersolitons to exist, the following conditions must be satisfied:

$$b < 0, \quad ac > 0, \quad ac < \frac{8}{27}b^2. \quad (36)$$

- (3) For a plasma that satisfies these criteria, supersolitons exist at velocities

$$v_{\min} < v < v_{\max}, \quad (37)$$

where

$$v_{\max} = v_+, \quad (38)$$

$$v_{\min} = \begin{cases} v_{DL} & \text{if } \frac{ac}{b^2} \leq \frac{5}{18}, \\ v_- & \text{if } \frac{5}{18} < \frac{ac}{b^2} < \frac{8}{27}, \end{cases} \quad (39)$$

$$v_{DL} = \frac{5b\left(\frac{5b^2 - 27ac}{27}\right) - 200\left(\frac{5b^2 - 18ac}{180}\right)^{3/2}}{27c^2}, \quad (40)$$

and

$$v_{\pm} = \frac{2b(16b^2 - 81ac) \pm 4\left(\frac{8b^2 - 27ac}{18}\right)^{3/2}}{27c^2}. \quad (41)$$

In (40), v_{DL} corresponds to the velocity of a double layer solution.

- (4) A comparison between the small-amplitude study and the analysis based on the fully nonlinear Sagdeev potential was performed. Based on that comparison, a region (in the compositional parameter space) for the existence of small-amplitude supersolitons was found. This region was established for plasma compositions very close to the supercritical plasma composition.

The mG equation can now be used to study collisions of overtaking supersolitons of small amplitudes.

IV. NON-INTEGRABILITY OF THE mG EQUATION

Some of the coefficients in (31) can be removed by scaling

$$t \rightarrow \alpha t, \quad \eta \rightarrow \beta \eta, \quad \Phi \rightarrow \gamma \Phi. \quad (42)$$

By choosing the parameters α , β , and γ appropriately, one obtains a normalized equation

$$\Phi_t + \frac{1}{2} \Phi_{\eta\eta\eta} \pm \Phi \Phi_{\eta} + d \Phi^2 \Phi_{\eta} \pm \Phi^3 \Phi_{\eta} = 0. \quad (43)$$

The signs in (43) and the choices of α , β , and γ depend on the signs of a , b , and c .

For the model under consideration, one can easily show¹⁹ that $A_4 = 81$ at the supercritical composition, so that $c = 2$. It can also easily be verified that $c > 0$ for plasma compositions near the supercritical composition. Based on the existence criteria (36), we restrict ourselves to compositions where $a > 0$, $b < 0$, and $c > 0$. Choosing the coefficients

$$\alpha = \left(\frac{c}{a}\right)^{3/4}, \quad \beta = \left(\frac{c}{a}\right)^{1/4}, \quad \gamma = -\sqrt{\frac{a}{c}} \quad (44)$$

yields

$$\Phi_t + \frac{1}{2} \Phi_{\eta\eta\eta} + \Phi \Phi_{\eta} + D \Phi^2 \Phi_{\eta} + \Phi^3 \Phi_{\eta} = 0, \quad (45)$$

with

$$D = -\sqrt{\frac{b^2}{ac}} = -\sqrt{\frac{3(15 - A_3)^2}{2(3 - A_2)(105 - A_4)}}. \quad (46)$$

To compute conservation laws of (45), we follow the approach of Verheest and Hereman²¹ which yields two conservation laws:

$$\Phi_t + \left(\frac{1}{2} \Phi^2 + \frac{D}{3} \Phi^3 + \frac{1}{4} \Phi^4 + \frac{1}{2} \Phi_{\eta\eta} \right)_{\eta} = 0, \quad (47)$$

and

$$(\Phi^2)_t + \left(\frac{2}{3} \Phi^3 + \frac{D}{2} \Phi^4 + \frac{2}{5} \Phi^5 + \Phi \Phi_{\eta\eta} - \frac{1}{2} \Phi_{\eta}^2 \right)_{\eta} = 0. \quad (48)$$

Using symbolic software developed by Poole and Hereman,²² an extensive search for polynomial conservation laws of (45) did not yield any additional results which suggests that (45) is not completely integrable. Equation (45) does not pass the Painlevé integrability test either as confirmed with the code of Baldwin and Hereman.²³ One should therefore not expect that solitary wave solutions of (45) would collide elastically and thus retain their shapes upon collisions.

V. SIMULATION OF SMALL-AMPLITUDE SUPERSOLITON COLLISIONS

We can now use the mG equation to simulate collisions between solitons and supersolitons. To do this, we construct a supersoliton solution and a slower soliton by numerically integrating the energy equation (34). The faster supersoliton solution is then shifted η_0 units to the left and added to the soliton solution. While the principle of superposition does not apply to nonlinear equations, it is assumed that the stability of the solutions ensures that the soliton and supersoliton propagation remain unaffected provided that the two solutions are sufficiently far apart.

It should be mentioned that the solutions must be constructed on a sufficiently large interval. Due to the instability of the energy integral (34), the numerical integration is not accurate enough to provide such solutions. We therefore applied the results from an asymptotic study²⁴ to construct sufficiently long tails for the solutions.

After constructing the appropriate initial potential Φ , the mG equation was integrated using a fourth-order Runge-Kutta method. We used finite differences to approximate the spatial derivatives and applied periodic boundary conditions. To avoid interference from the periodic boundary assumption, we had to choose a sufficiently large interval length.

The simulations reveal that the supersoliton breaks up during the collision, so that only regular solitons emerge after the collision. To illustrate this, we discuss a typical result obtained from simulations with $D = -\sqrt{3.6}$. The initial disturbance consists of a supersoliton with velocity $v = 0.5(v_{DL} + v_{\max}) \approx 0.1218$ that is shifted $\eta_0 = 75$ units to the left, and a slower soliton with velocity $v = 0.1$. For this simulation, the interval length is $L = 1200$ and a grid with $N = 25\,600$ points are used. Therefore, the spatial width is $\Delta\eta \approx 0.047$. An integration increment of $\Delta t = 10^{-4}$ is used.

The results are shown in Fig. 1, where the magnitude of the solution Φ is plotted as a function of η and t . Here, we see that the supersoliton (initially on the left) widens around $t \approx 1000$, before the collision takes place. The fact that the supersoliton breaks up during this time is not obvious from the figure. The amplitude of the collision peaks around $t = 3600$, before two solitons with smaller amplitudes emerge. It is therefore clear that the collision is inelastic.

To see the breaking up of the supersoliton more clearly, in Fig. 2 we graphed the η profiles of the electric field, $E = -\partial\Phi/\partial\eta$, at different values of t . In panel (a) of Fig. 2, the initial condition is shown. The characteristic ‘‘wiggles’’ of the supersoliton are clearly visible to the left of the regular soliton. As the supersoliton approaches the regular soliton, the supersoliton starts to deform. This is shown in panel (b)

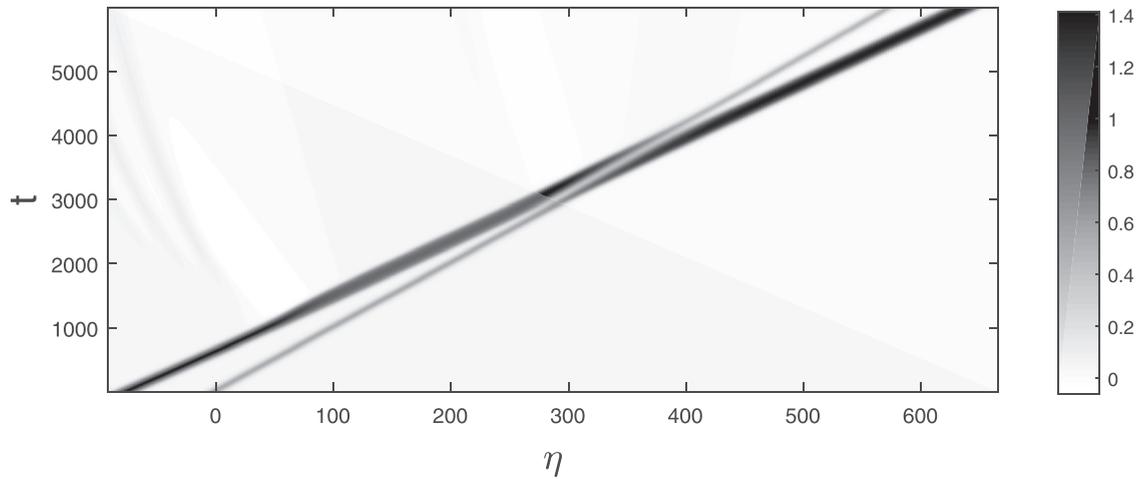


FIG. 1. Simulation of a supersoliton overtaking a regular soliton. The electrostatic potential Φ is plotted as a function of η and t .

for $t = 1100$. The supersoliton breaks up to form a regular soliton. Panel (c) shows the solution at $t = 1400$, after the supersoliton deformed to become a regular soliton.

The collision of the resulting two solitons is shown in panels (d)–(f) of Fig. 2. The faster soliton overtakes the slower, resulting in a transient solution as shown in panel (d) for $t = 3000$. Eventually, the faster soliton re-emerges in front of the slower one, as shown in panel (e) for $t = 4400$. Beyond $t = 4400$, the separation between the two solitons increases, as depicted in panel (f) for $t = 5500$.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we applied reductive perturbation analysis to study small-amplitude supersolitons in a plasma consisting of cold ions and two-temperature Boltzmann electrons. To do so, we considered near-supercritical plasma compositions. We derived a generalized Korteweg-de Vries equation, referred to as the modified Gardner equation. For that equation, we derived the necessary conditions for small-amplitude supersolitons to exist.

We also used the equation to study the collision properties of small-amplitude supersolitons, both theoretically and through simulations. Theoretically, we showed that in contrast to the KdV and mKdV equations, the mG equation is not completely integrable. Hence, collisions of small-amplitude supersolitons will be inelastic. Numerical simulations of the collisions between solitons and supersolitons show that the supersolitons break up during the collision to form a regular soliton. This is very different from elastic collisions of regular solitons.

These results show that, in the small-amplitude regime, supersolitons are not as robust as regular solitons and may break up during collisions. This suggests that their life spans may be much shorter than that of regular solitons and might explain the low number of supersoliton observations in space plasmas.

However, caution must be taken in the interpretation of these results. Indeed, for these conclusions to be valid, our results must be extended beyond the small-amplitude regime. To do so, one has to study the collision properties of supersolitons in laboratory experiments or numerical simulations. In

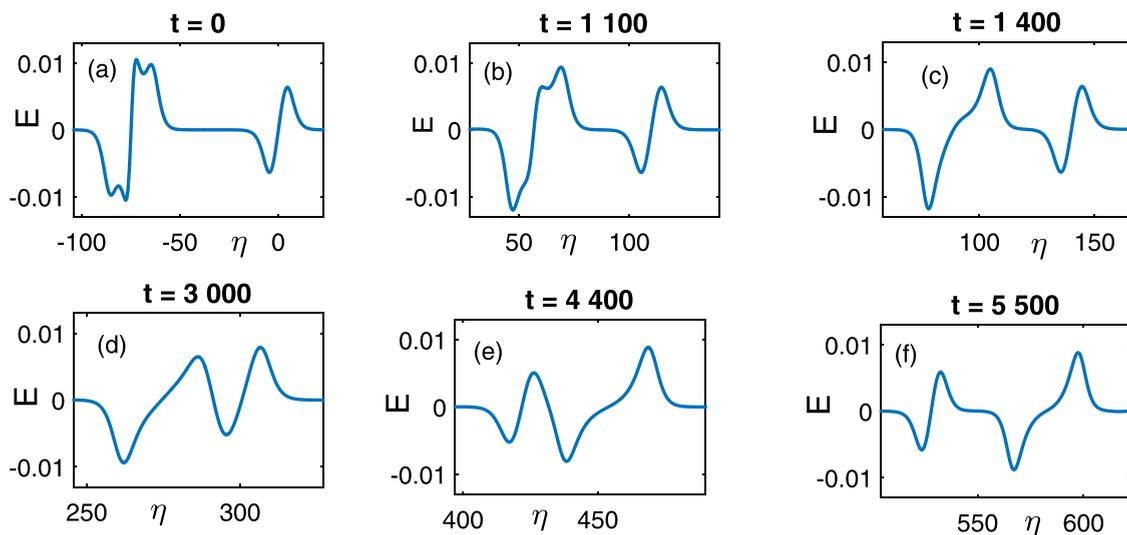


FIG. 2. Simulation of supersoliton overtaking a regular soliton. The electric field E is shown at different times t , as specified on top of each panel.

addition, head-on collisions lie beyond the scope of this analysis.

In conclusion, we hope that our results will generate interest in the topic of supersoliton collisions, and that this study can be used as a benchmark for further investigations.

ACKNOWLEDGMENTS

C.P.O. wishes to acknowledge the financial assistance of the National Research Foundation (NRF) towards this research. Opinions expressed and conclusions arrived at are those of the authors and are not necessarily to be attributed to the NRF.

- ¹A. E. Dubinov and D. Yu. Kolotkov, *Plasma Phys. Rep.* **38**, 909 (2012).
- ²A. E. Dubinov and D. Yu. Kolotkov, *High Energy Chem.* **46**, 349 (2012).
- ³A. E. Dubinov and D. Yu. Kolotkov, *IEEE Trans. Plasma Sci.* **40**, 1429 (2012).
- ⁴F. Verheest, M. A. Hellberg, and I. Kourakis, *Phys. Plasmas* **20**, 012302 (2013).
- ⁵F. Verheest, M. A. Hellberg, and I. Kourakis, *Phys. Rev. E* **87**, 043107 (2013).
- ⁶F. Verheest, M. A. Hellberg, and I. Kourakis, *Phys. Plasmas* **20**, 082309 (2013).
- ⁷S. K. Maharaj, R. Bharuthram, S. V. Singh, and G. S. Lakhina, *Phys. Plasmas* **20**, 083705 (2013).
- ⁸C. P. Olivier, S. K. Maharaj, and R. Bharuthram, *Phys. Plasmas* **22**, 082312 (2015).
- ⁹F. Verheest and C. P. Olivier, *Phys. Plasmas* **24**, 113708 (2017).
- ¹⁰A. E. Dubinov and D. Yu. Kolotkov, *Rev. Mod. Plasma Phys.* **2**, 2 (2018).
- ¹¹M. Temerin, K. Cerny, W. Lotko, and F. S. Mozer, *Phys. Rev. Lett.* **48**, 1175 (1982).
- ¹²M. H. Boehm, C. W. Carlson, J. McFadden, and F. S. Mozer, *Geophys. Res. Lett.* **11**, 511, <https://doi.org/10.1029/GL011i005p00511> (1984).
- ¹³S. R. Bounds, R. F. Pfaff, S. F. Knowlton, F. S. Mozer, M. A. Temerin, and C. A. Kletzing, *J. Geophys. Res.* **104**, 28709, <https://doi.org/10.1029/1999JA900284> (1999).
- ¹⁴J. S. Pickett, L.-J. Chen, O. Santolik, S. Grimald, B. Lavraud, O. P. Verkhoglyadova, B. T. Tsurutani, B. Lefebvre, A. Fazakerley, G. S. Lakhina, S. S. Ghosh, B. Grison, P. M. E. Décréau, D. A. Gurnett, R. Torbert, N. Cornilleau-Wehrin, I. Dandouras, and E. Lucek, *Nonlinear Processes Geophys.* **16**, 431 (2009).
- ¹⁵A. Kakad, A. Lotekar, and B. Kakad, *Phys. Plasmas* **23**, 110702 (2016).
- ¹⁶R. V. Sagdeev, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, 1966), Vol. 4, pp. 23–91.
- ¹⁷H. Washimi and T. Taniuti, *Phys. Rev. Lett.* **17**, 996 (1966).
- ¹⁸F. Verheest and M. A. Hellberg, *Phys. Plasmas* **22**, 012301 (2015).
- ¹⁹F. Verheest, C. P. Olivier, and W. A. Hereman, *J. Plasma Phys.* **82**, 905820208 (2016).
- ²⁰C. P. Olivier, F. Verheest, and S. K. Maharaj, *J. Plasma Phys.* **83**, 905830403 (2017).
- ²¹F. Verheest and W. A. Hereman, *Phys. Scr.* **50**, 611 (1994).
- ²²D. Poole and W. Hereman, *J. Symbolic Comput.* **46**, 1355 (2011).
- ²³D. Baldwin and W. Hereman, *J. Nonlinear Math. Phys.* **13**, 90 (2006).
- ²⁴C. P. Olivier, F. Verheest, and S. K. Maharaj, *J. Plasma Phys.* **83**, 905830605 (2017).