Classes of Dunford-Pettis-type operators with applications to Banach spaces and Banach lattices

ED Zeekoei

20485379

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NORTH-WEST UNIVERSITY
YUNIBESITI YA BOKONE-BOPHIRIMA
NOORDWES-UNIVERSITEIT

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Abstract

The aim of this study is to extend existing knowledge of Dunford-Pettis operators on Banach spaces and Banach lattices and their variants. Using the concept of p-convergent operators as basis, we introduce the Dunford-Pettis-type operators by introducing the so-called weak p-convergent, disjoint p-convergent as well as the weak* p-convergent operators on Banach spaces and Banach lattices. The concept of the DP^*P_p on Banach spaces, which was introduced in the paper [37], leads to the introduction of the concept of weak* p-convergent operator in this study (and in the paper [67]). An interesting fact is that for the Banach spaces, most of the existing results in the study field of Dunford-Pettis operators and their variants may be carried over in our context where we consider the weak p-summable sequences and base most of our definitions and results on these types of sequences. However, the Banach lattice cases are a bit trickier and we have to revert to crafty plans in order to find proofs for our results. We also study the coincidences of these types of operators as well as a type of domination property that each of these types of operators possess. A classic property is that of the Schur property, which is used to characterise the almost Dunford-Pettis operators. Associated with this is the so-called positive Schur property of Banach lattices. We follow these concepts and develop the so-called Schur property of order p as well as its positive version. A discussion of sequentially p-limited operators introduced by Karn and Sinha (see [48]) to study the p-DPP, follows. Motivated by the Banach ideal property of $(Lt_p, \ell t_p)$, we introduce the general concept of "operator $[Y, p]$-summable sequence in a Banach space X, consider the vector space $Y_p(X)$ of all operator $[Y, p]$-summable sequences in X and then introduce a norm on the space. $Y_p(X)$ turns out to be a Banach space and we apply the results of the general setting to the special setting of operator p-summable sequences in a Banach space X. This leads to the extension and improvement of results in [48]. We then apply the concept of a disjoint p-convergent operator to introduce the so-called disjoint p-convergent functions on Banach lattices. Our specific focus here is to establish under what conditions continuous n-homogeneous polynomials, holomorphic maps and symmetric separately compact bilinear maps are disjoint p-convergent functions.

Keyterms: p-Convergent operator, Disjoint p-convergent operator, Weak p-convergent operator, Schur property of order p, Positive Schur property of order p, weak* p-convergent operator, p-Gelfand-Phillips property, Operator p-summable sequence, Sequentially p-limited operator, Disjoint Dunford-Pettis property of order p, Disjoint Dunford-Pettis* property of order p, p-Convergent function, Disjoint p-convergent function.

Contents

Introduction

The literature contains an extensive theory of classes of operators acting on relatively weakly compact subsets and mapping these subsets to either norm totally bounded subsets, Dunford-Pettis subsets or limited subsets. Variations of these types of operators have been developed and applied extensively in the study of the geometrical properties of Banach spaces and Banach lattices.

We recall the notion of p-convergent operators and introduce the notions of operators that are weak p-convergent and weak* p-convergent between Banach spaces, where $1 \leq p < \infty$. We also consider the notions of disjoint p-convergent operators from a Banach lattice to a Banach space, positive weak p-convergent and positive weak* p-convergent operators between Banach lattices, with $1 \leq p < \infty$. These operators are either weaker or stronger variants of the Dunford-Pettis operators, which we label as "Dunford-Pettis-type operators", for obvious reasons. Corresponding to these classes of operators are therefore weaker or stronger variants of the Dunford-Pettis property on Banach spaces.

Note that we may regard the Dunford-Pettis operators and its variants, the weak Dunford-Pettis, almost Dunford-Pettis and weak* Dunford-Pettis operators as the ∞-convergent, weak ∞-convergent, almost ∞-convergent and weak* ∞-convergent operators respectively.

As in the case when $p = \infty$, we investigate when these different variants of operators coincide with each other and study their domination properties when working in the context of Banach lattices. We then study variants of the Schur and positive Schur properties and use them to characterise various Dunford-Pettis-type operators as well as establishing coincidences between these types of operators.

We next conduct a study of sequentially limited operators. In a brief discussion of sequentially p-limited operators, Karn and Sinha introduce a norm ℓt_p on each vector space $Lt_p(X,Y)$ of sequentially p-limited operators as X,Y run through the family of all Banach spaces and show that $(Lt_p, \ell t_p)$ is a normed operator ideal. Inspired by the Banach ideal property of $(Lt_p, \ell t_p)$, we introduce the general concept of "operator $[Y,p]$-summable sequence" in a Banach space X, consider the vector space $Y_p(X)$ of all operator $[Y,p]$-summable sequences in X and introduce a norm on this space. We prove that $Y_p(X)$ is a Banach space. We apply the results of the general setting to the special setting of operator p-summable sequences in a Banach space X.

Finally, we apply our study of Dunford-Pettis-type operators to the so-called Dunford-Pettis-type functions. We first explore elementary properties of polynomials and holomorphic functions found in the literature. We then introduce the so-called disjoint Dunford-Pettis property of order p and the disjoint Dunford-Pettis* property of order p, and characterise these two properties in terms of sequences. The disjoint Dunford-Pettis property of order p is then also characterised in terms of disjoint p-convergent operators. Following the definition of a p-convergent function in [37], we introduce the so-called disjoint p-convergent functions. We then provide conditions under which a polynomial, holomorphic function and a symmetric separately compact bilinear map are disjoint p-convergent.

The thesis consists of six chapters of which content we explore next:

In Chapter 1 we provide the reader with some basic facts on Banach spaces and Banach lattices. Here we list the most important definitions and summarise the most important results from the literature, which will be of use in later chapters.

In Chapter 2 we provide an extensive review of existing literature on the Dunford-Pettis operators and its variants.

In Section 2.1 we review the Dunford-Pettis operators and consider some well-known facts with regards to Dunford-Pettis operators which include:

(i) Every Dunford-Pettis operator is continuous.

(ii) A compact operator is necessarily a Dunford-Pettis operator.

(iii) If X is a reflexive Banach space, then an operator with domain X is Dunford-Pettis if and only if it is compact.

(iv) A Dunford-Pettis operator need not be a compact operator.

The question whether a positive operator S between Banach lattices is dominated by a Dunford-Pettis operator, say T, would S then be necessarily a Dunford-Pettis operator, has become commonplace in the literature and we consider two results of this so-called domination-property of the Dunford-Pettis operators.

In Section 2.2 we explore the class of weak Dunford-Pettis operators, which have the following properties:

(i) Every Dunford-Pettis operator is a weak-Dunford-Pettis operator.

(ii) If Y is reflexive, then the notions of Dunford-Pettis and weak Dunford-Pettis operator coincide.

(iii) If Y has the Dunford-Pettis property, then every continuous operator from X to Y is a weak Dunford-Pettis operator.

(iv) A weak Dunford-Pettis operator need not be a Dunford-Pettis operator.

When considered on Banach lattices, these classes of operators also exhibit a domination-property in that if a positive operator S is dominated by a weak Dunford-Pettis operator, then S is also a weak Dunford-Pettis operator.

Section 2.3 focuses on the so-called almost Dunford-Pettis operators as well as an important property on Banach lattices, which is the positive Schur property.

The positive Schur property characterises the almost Dunford-Pettis operators and plays a role in establishing the coincidence of positive weak Dunford-Pettis operators and almost Dunford-Pettis operators.

We also consider the coincidence of positive weak Dunford-Pettis and almost Dunford-Pettis operators by reviewing a series of results. We then define the Schur property which is used to characterise the Dunford-Pettis operators as well as to investigate the coincidence between the weak Dunford-Pettis and the Dunford-Pettis operators.

In Section 2.4 we conclude our literature review with an exposition of the so-called weak* Dunford-Pettis operators.

Some of the most important characteristics of this class of operators include the following:

(a) The class of w*DP operators is bigger than the class of Dunford-Pettis operators, but smaller than the class of weak Dunford-Pettis operators.

(b) id_{ℓ_∞} is w*DP (since ℓ_∞ has DP* property), but is not a Dunford-Pettis operator (since ℓ_∞ does not have the Schur property).

(c) id_{c_0} is weak Dunford-Pettis (since c_0 has the DP property), but is not w*DP (since c_0 does not have the DP* property).

(d) If Y is a Grothendieck space, then the notions of weak Dunford-Pettis and w*DP operators coincide.

A bounded linear operator T from a Banach space X into another Y is said to be a limited operator if it carries the unit ball in X, $\mathcal{B}_X$ into a limited set of Y. An operator T is limited if and only if T^* takes weak* null sequences to norm null ones. Furthermore, a bounded linear operator $T : X \to Y$ between two Banach spaces is said to be limited completely continuous (lcc for short) if it carries sequences in X which are both limited and weakly null to norm null sequences in Y. We then provide from the literature a characterisation of the lcc in terms of Banach lattices and conclude with an exposition of the domination property of the positive weak* Dunford-Pettis operators.

In Chapter 3 we explore p-convergent operators and its variants, on Banach spaces and Banach lattices. This is an extension of the existing knowledge of Dunford-Pettis operators and almost Dunford-Pettis operators. The main results of this chapter have been submitted for publication (cf. [68]).

In Section 3.1 we introduce the so-called weak p-convergent operators, the disjoint p-convergent operators and the positive Schur property of order

p.

The family of p-convergent operators was introduced in the paper [20] and applied in the paper [37] to study the DP^*-property of order p in Banach spaces.

In the context of Banach lattices we consider weakly p-summable sequences with disjoint elements for which the sequence of moduli are still weakly p-summable, and show that this implies that the sequences of positive and negative parts are again weakly p-summable. Working now in this "disjoint" setting, we define the so-called disjoint p-convergent operators.

It turns out that this class of operators satisfies a "domination" property, as was the case with the almost Dunford-Pettis operators. Motivated by the notion of a weak Dunford-Pettis operator, we define the so-called weak p-convergent operators. We are then able to describe a relationship between weak p-convergent and p-convergent operators:

Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. The following statements are equivalent:

(i) T is weak p-convergent.

(ii) ST is p-convergent for each weakly compact operator $S : Y \to Z$ and any Banach space Z.

(iii) ST is p-convergent for each weakly compact operator $S : Y \to c_0$.

Clearly, each p-convergent operator is weak p-convergent and it follows that if Y is a reflexive Banach space, then each weak p-convergent operator from any Banach space to Y is p-convergent.

Next we introduce the notion of "positive Schur property of order p" (abbreviated as SP_p^+) and agree that E has the SP_∞^+ if each sequence $(x_n) \in c_0^{weak}(E)$ with positive terms, is norm convergent to 0. If we assume that $1 \leq p \leq \infty$, then SP_∞^+ will coincide with the well-known positive Schur property. With this in mind, we then characterise the SP_p^+ in terms of disjoint weakly p-summable sequences by proving that a Banach lattice E has the positive Schur property of order p, if and only if each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, is norm convergent to 0. It turns out that this is equivalent to id_E being disjoint p-convergent.

In order to extend this, as to include more operators, we propose a more general characterisation of the SP_p^+:

Let E be a Banach lattice and $1 \leq p < \infty$. Then, the following statements are equivalent:

(1) Each positive operator from E to ℓ_∞ is disjoint p-convergent.

(2) E has the SP_p^+.

Most of the results concerning the positive Schur property can be carried over to the setting of the positive Schur property of order p. Some of these results include:

(A) Let E and F be two Banach lattices such that F is a dual Banach lattice. Then the following assertions are equivalent:

   (1) Each positive weak p-convergent operator $T : E \to F$ is disjoint p-convergent.

   (2) One of the following assertions is valid:

      (a) E has the SP_p^+.

      (b) F is a KB-space.

(B) Let E and F be two Banach lattices with F Dedekind σ-complete. If each positive weak p-convergent operator $T : E \to F$ is disjoint p-convergent, then one of the following assertions is valid:

   (1) E has the SP_p^+.

   (2) F has an order continuous norm.

(C) Let E and F be two Banach lattices with F Dedekind σ-complete. If the norm of F is not order continuous, then the following assertions are equivalent:

   (1) Each positive operator $T : E \to F$ is disjoint p-convergent.

   (2) Each positive weak p-convergent operator $T : E \to F$ is disjoint p-convergent.

   (3) E has the SP_p^+.

In Section 3.2 we discuss the positive weak p-convergent and its relationship to the p-convergent operators.

It is well-known that the lattice operations in AM-spaces are weakly sequentially continuous. However, in the spaces $L_p[0,1]$ (where $1 \leq p < \infty$) the lattice operations fail to be weakly sequentially continuous. Since we need the lattice operations to satisfy a seemingly weaker property than being weakly sequentially continuous, we then introduce the notion "weakly sequentially p-continuous".We also introduce the so-called Schur property of order p. In general, the weak sequential continuity of the lattice operations in a Banach lattice is not implied by the weakly sequentially p-continuity of the same. For instance, the space $L_1[0,1]$ has SP_1; thus, the lattice operations in $L_1[0,1]$ are weakly sequentially 1-continuous, but they are not weakly sequentially continuous.

The relationship between the positive weak p-convergent and p-convergent operators, are formulated as follows:

Let E and F be two Banach lattices. Then each positive weak p-convergent operator from E into F is p-convergent if one of the following assertions is valid:

(1) F is a dual KB-space and the lattice operations in E are weakly sequentially p-continuous.

(2) F is a discrete KB-space.

(3) The norm of the topological bi-dual F^{**} is order continuous and the lattice operations in E are weakly sequentially p-continuous.

(4) E has the SP_p.

(5) F is reflexive.

In Section 3.3 we further explore the Schur property of order p on Banach lattices.

We first consider necessary conditions for the domination property of positive p-convergent operators on Banach lattices to hold. Now, if a Banach lattice E is an AM-space with unit, then $(|x_i|) \in \ell_p^{weak}(E)$ for each

$(x_i) \in \ell_p^{weak}(E)$.

In this section we also introduce the notion of a weak p-consistent Banach lattice E as a lattice such that if $(x_i) \in \ell_p^{weak}(E)$ then $(|x_i|) \in \ell_p^{weak}(E)$, for $1 \leq p < \infty$.

Clearly if a Banach lattice E has the SP_p, then the lattice operations are in particular weakly sequentially p-continuous and E has the SP_p^+. On the other hand, if E is a weak p-consistent Banach lattice (for instance an AM-space with unit) and E has the SP_p^+, then for each $(x_n) \in \ell_p^{weak}(E)$ we have $(|x_n|) \in \ell_p^{weak}(E)$ and so $\|x_n\| = \||x_n|\| \to 0$ as $n \to \infty$. This says that E has the SP_p.

Using the notion of weak p-consistent, we formulate the following proposition:

Let E be a weak p-consistent Banach lattice and F any Banach lattice. If $S, T : E \to F$ are positive operators satisfying $0 \leq S \leq T$ and T is p-convergent, then likewise S is p-convergent.

If $T : E \to F$ is a bounded, linear operator between two Banach lattices and the target space is an AL-space, then T is p-convergent if and only if $|Tx_n| \to 0$ as $n \to \infty$ weakly in F for all weakly p-summable sequences (x_n).

Using this characterisation, we may apply the theorem to positive p-convergent operators:

Let E be a Banach lattice and let F be an AL-space in which the lattice operations are weakly sequentially p-continuous, then each positive operator $T : E \to F$ is p-convergent.

We obtain a similar characterisation of the SP_p as we have for the SP_p^+:

Let E be a Banach lattice. Then, the following assertions are equivalent:

(1) Each positive operator from E into ℓ_∞ is p-convergent.

(2) E has the SP_p.

This theorem is critical in proving the following coincidence of positive weak p-convergent and p-convergent operators:

Let E and F be two Banach lattices with F Dedekind σ-complete. If each positive weak p-convergent operator from E into F is p-convergent, then one of the following assertions is valid:

(1) E has the SP_p.

(2) F has an order continuous norm.

We then show that in order for a Banach space to have the SP_p, it is sufficient for the space to be separable and have the DP^*P_p.

The following observation regarding p-convergent operators and the SP_p is made:

Let E and F be two Banach lattices such that E is separable and F is not a KB-space. Then the following assertions are equivalent:

(1) Each bounded linear operator $T : E \to F$ is p-convergent.

(2) E has the SP_p.

When E and F are two Banach lattices such that E is a Gelfand-Phillips space, then if each weak p-convergent operator $T : E \to F$ is p-convergent, then one of the following assertions is valid:

(1) E has the SP_p.

(2) F is a KB-space.

Finally we obtain the following:

Let E and F be two Banach lattices such that E has an order continuous norm and F is not a KB-space. Then the following assertions are equivalent:

(1) Each operator $T : E \to F$ is p-convergent.

(2) E has the SP_p.

In Chapter 4 we introduce the weak* p-convergent operators on Banach spaces and Banach lattices. The content of the chapter is an extension of work done on the weak Dunford-Pettis operators. The main results of this chapter has been accepted for publication (cf. [67]).

We introduce and study the notion of "weak* p-convergent operator", and discuss the relationship between the weak* p-convergent operators and the p-convergent operators, which plays an important role in the study of the DP^*-property of order p. Some new characterizations of Banach spaces with the DP^*-property of order p are obtained, the p-Gelfand-Phillips property is introduced and the behaviour of weak* p-convergent operators on Banach spaces with this property (with focus on Banach lattices with the p-Gelfand-Phillips property) is investigated. We then consider the domination properties of positive p-convergent and positive weak* p-convergent operators on Banach lattices.

In Section 4.1 we introduce the so-called weak* p-convergent operators, their role in the study of the DP^*P_p as well as the so-called p-Gelfand-Phillips space in contexts of both Banach spaces and Banach lattices.

We may connect this class of operators with the class of p-convergent operators as follows:
Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. The following are equivalent:

(a) T is weak* p-convergent.

(b) ST is p-convergent for each $S \in \mathcal{L}(Y,Z)$ and any separable Banach space Z.

(c) ST is p-convergent for each $S \in \mathcal{L}(Y,c_0)$.

It follows that if X and Y are Banach spaces, with Y separable, then each weak* p-convergent operator $T : X \to Y$ is p-convergent. Moreover, if X and Y are Banach spaces, with Y a separable reflexive space, then the families of p-convergent operators, weak p-convergent operators and weak* p-convergent operators from X to Y coincide.

In [37] we showed that a Banach space X has DP^*P_p if and only if $\langle x_n^*, x_n \rangle \to 0$ as $n \to \infty$ for all weak* null sequences $(x_n^*) \subset X^*$ and all weakly p-summable sequences $(x_n) \subset X$. It turns out that a Banach space X has DP^*P_p if and only if the identity operator id_X is weak* p-convergent. If X is separable, then id_X is p-convergent.

The following are well-known facts:

(1) A sequence (x_n) in X is limited if and only if $x_n^*(x_n) \to 0$ for each weak* null sequence (x_n^*) in X^*.

(2) X is a Gelfand-Phillips space if and only if every limited weakly null sequence in X is norm null.

With (1) and (2) in mind, we introduce the following definition of the so-called p-Gelfand-Phillips property:

Let $1 \leq p < \infty$. A Banach space X is said to have the p-Gelfand-Phillips property (p-GPP for short) if every limited weakly p-summable sequence (x_n) in X is norm null. If X has this property, then we call X a p-Gelfand-Phillips space. For the case of $p = \infty$, we consider the ∞-Gelfand-Phillips property the same as the Gelfand-Phillips property (GPP for short). Clearly, if $1 \leq p < q$, then each limited weakly p-summable sequence in a Banach space X will also be a limited weakly q-summable sequence and so the q-GPP will imply the p-GPP on X and they will be implied by the GPP. Some classical Banach spaces, such as c_0 and ℓ_1 have the GPP and thus they also have the p-GPP for all $1 \leq p < \infty$. It is known that ℓ_∞ lacks the GPP.

Motivated by the definition of limited completely continuous operators, we introduce the limited p-convergent operators:

A bounded linear operator $T : X \to Y$ between two Banach spaces is called limited p-convergent if it carries limited weakly p-summable sequences in X to norm null ones in Y. It is clear that a Banach space X has the p-GPP if and only if id_X is limited p-convergent. By definition, p-convergent operators are limited p-convergent. If Y is separable, then weak* p-convergent operators with target space Y are limited p-convergent. In particular, the identity operators on separable spaces with the DP^*P_p are limited p-convergent, i.e. separable spaces with the DP^*P_p are p-Gelfand-Phillips spaces.

The following condition on the underlying Banach lattices ensures that weak* p-convergent operators are limited p-convergent:

Let E and F be Banach lattices such that F is Dedekind σ-complete. Then the following assertions are equivalent:

(1) Each weak* p-convergent operator T from E into F is limited p-convergent.

(2) Either E has the p-GPP or F has an order continuous norm.

This has the following consequences:

(a) Let E be a Banach lattice. Then the following assertions are equivalent:

   (1) Each weak* p-convergent operator T from E into ℓ_∞ is limited p-convergent.

   (2) E has the p-GPP.

(b) Let F be a σ-Dedekind complete Banach lattice. Then the following assertions are equivalent:

   (1) Each weak* p-convergent operator T from ℓ_∞ into F is limited p-convergent.

   (2) F has order continuous norm.

In Section 4.2 we consider the domination properties of the positive weak* p-convergent operators.

The fact that every disjoint sequence in the solid hull of a relatively weakly compact subset of a Banach lattice converges weakly to zero, plays an important role in the proofs of many results concerning Dunford-Pettis operators and the Dunford-Pettis property on Banach lattices. In the context of this chapter, we formulate the following lemma, that concerns Banach lattices with non-trivial type:

Let E be a Banach lattice with type q (with $1 < q \leq 2$) and let $p \geq q'$. Each disjoint sequence (x_n) in the solid hull of a relatively weakly compact subset W of E belongs to $\ell_p^{weak}(E)$.

Using this and the well-known Kalton-Saab Theorem which states that if a positive operator $S : E \to F$ between two Banach lattices (where F has order continuous norm) is dominated by a Dunford-Pettis operator, then S itself is Dunford-Pettis, we then formulate:

Let E, F be Banach lattices such that E has type $1 < q \leq 2$ and F has order continuous norm. If $T : E \to F$ is a positive p-convergent operator, where $p \geq q'$, then each positive operator $S : E \to F$ satisfying $0 \leq S \leq T$ is p-convergent itself.

In order to study the domination property for the class of weak* p-convergent operators, we need the following result:

Let E, F be Banach lattices such that E has type $1 < q \leq 2$ and F is σ-Dedekind complete. Let $T : E \to F$ be a positive weak* p-convergent operator. Then for every weakly p-summable sequence (x_n) in E^+ and every weak* null sequence (f_n) in F^*, we have

$$|f_n|(Tx_n) \to 0 \text{ as } n \to \infty.$$

By the definition of weak* p-convergent operator, it follows that for sequences (x_n) and (f_n) we have $f_n(Tx_n) \to 0$ as $n \to \infty$. The important consequence of the σ-Dedekind completeness of F is that we have the stronger property $|f_n|(Tx_n) \to 0$ as $n \to \infty$. The role of the σ-Dedekind completeness of F is to assure that both the sequences of positive parts and absolute values of a disjoint weak* null sequence in F^* are weak* null themselves. Using these facts, we formulate:

Let $T : E \to F$ be a positive weak* p-convergent operator (for $1 \leq p < \infty$), where E, F are Banach lattices such that E is weak p-consistent and F is σ-Dedekind complete. If $0 \leq S \leq T$, then S is weak* p-convergent.

From this it follows that if $T : E \to F$ is a positive weak* p-convergent operator (for $1 \leq p < \infty$), with E and F Banach lattices, such that E is an AM-space with unit and F is σ-Dedekind complete. Then if $0 \leq S \leq T$, S is weak* p-convergent.

In conclusion, as the condition "E is weak p-consistent" is restrictive, we formulate the following version:

Let E, F be Banach lattices such that E has type $1 < q \leq 2$ and F is σ-Dedekind complete. If $T : E \to F$ is a positive weak* p-convergent operator, where $p \geq q'$, then each positive operator $S : E \to F$ satisfying $0 \leq S \leq T$ is weak* p-convergent itself.

In Chapter 5 we study the sequentially limited operators on Banach spaces. The main results of this chapter has been published the previous year (cf. [38]).

In Section 5.1 we introduce the so-called operator $[Y, p]$-summable sequences:

Let X, Y be given Banach spaces and let $1 \leq p < \infty$. A sequence (x_n) in X is called operator $[Y, p]$-summable if $\sum\limits_{n=1}^{\infty} \|Tx_n\|^p < \infty$ for all $T \in \mathcal{L}(X, Y)$, i.e. if $(Tx_n) \in \ell_p^{strong}(Y)$ for all $T \in \mathcal{L}(X, Y)$.

Let $Y_p(X) := \{(x_i) \in X^{\mathbb{N}} : (x_i) \text{ is operator } [Y, p]\text{-summable}\}$. For a given $(x_i) \in Y_p(X)$, we define an operator $\Theta : \mathcal{L}(X, Y) \to \ell_p^{strong}(Y) : T \mapsto (Tx_n)$. It turns out that this operator is bounded and linear. Furthermore, $\sup \left\{ \left(\sum\limits_{n=1}^{\infty} \|Tx_n\|^p \right)^{1/p} : T \in \mathcal{L}(X, Y), \|T\| \leq 1 \right\} = \|\Theta\| < \infty$. Since this is true for each $(x_i) \in Y_p(X)$, we define $\| \cdot \|_{Y_p} : Y_p(X) \to \mathbb{R}$ by $\|(x_i)\|_{Y_p} := \sup \left\{ \left(\sum\limits_{n=1}^{\infty} \|Tx_n\|^p \right)^{1/p} : T \in \mathcal{L}(X, Y), \|T\| \leq 1 \right\}$, where $\| \cdot \|_{Y_p}$ defines a norm on the vector space $Y_p(X)$. We then show that $(Y_p(X), \| \cdot \|_{Y_p})$ is a Banach space.

If we replace Y in Definition 5.1.1 by ℓ_p, instead of "operator $[\ell_p, p]$-summable", we use the phrase "operator p-summable" and recall that a sequence (x_n) is called operator p-summable, if $(Tx_n) \in \ell_p^{strong}(\ell_p)$ for all $T \in \mathcal{L}(X, \ell_p)$.

If we let $\ell_p^o(X)$ denote the vector space of all operator p-summable sequences in the Banach space X, then for $(x_i) \in \ell_p^o(X)$ we let

$\|(x_i)\|_p^o := \sup\left\{\left(\sum_n \|Tx_n\|_p^p\right)^{1/p} : T \in \mathcal{L}(X, \ell_p), \|T\| \leq 1\right\}$, then it follows that $(\ell_p^o(X), \|\cdot\|_p^o)$ is a Banach space. Let $1 \leq p < \infty$.

Recall that an operator $T \in \mathcal{L}(X,Y)$ is said to be p-summing if $(Tx_n) \in \ell_p^{strong}(Y)$ for all $(x_n) \in \ell_p^{weak}(X)$. The vector space $\Pi_p(X,Y)$ of all p-summing operators is a Banach space with respect to the norm $\pi_p(T) := \sup\{\|(Tx_n)\|_p : \|(x_n)\|_p^{weak} \leq 1\}$. A Banach space X is called a weak p-space (or X is said to have the p-Dunford-Pettis property) if $\ell_p^o(X) = \ell_p^{weak}(X)$. This is the case if and only if $\Pi_p(X, \ell_p) = \mathcal{L}(X, \ell_p)$. It is therefore immediately clear that ℓ_p itself is not a weak p-space.

In Section 5.2 we consider the so-called sequentially p-limited operators which maps weakly p-summable sequences to operator p-summable sequences, i.e.

Let $1 \leq p < \infty$. An operator $T \in \mathcal{L}(X,Y)$ is called sequentially p-limited if $(Tx_n) \in \ell_p^o(Y)$ for all $(x_n) \in \ell_p^{weak}(X)$.

It is clear that id_X is sequentially p-limited if and only if X is a weak p-space. An operator $T : X \to Y$ is sequentially p-limited if and only if RT is p-summing for all $R \in \mathcal{L}(Y, \ell_p)$. We let $Lt_p(X,Y) := \{T \in \mathcal{L}(X,Y) : T \text{ is sequentially } p\text{-limited}\}$ and define a norm on $Lt_p(X,Y)$ by $\ell t_p(T) := \sup\{\pi_p(RT) : R \in \mathcal{L}(Y, \ell_p) \text{ and } \|R\| \leq 1\}$ and then the pair $(Lt_p, \ell t_p)$ is a normed operator ideal. Moreover, we show that $(Lt_p, \ell t_p)$ is a Banach operator ideal.

Let $1 < p < \infty$, then it is easily verified that if the second dual operator $T^{**} : X^{**} \to Y^{**}$ of the operator $T \in \mathcal{L}(X,Y)$ is sequentially p-limited, then so is T. Moreover, we also have:

Let $1 < p < \infty$. If an operator $T : X \to Y$ is sequentially p-limited and weakly compact, then so is its second dual T^{**}.

Since not all sequentially p-limited operators are weakly compact, we formulate a lemma that lists seven conditions from the literature under which each operator $T \in \mathcal{L}(X,Y)$ is weakly compact, hence when each sequentially

p-limited operator is weakly compact. Furthermore, if the Banach spaces X and Y satisfy any one of these seven conditions, then if $T : X \to Y$ is sequentially p-limited, so is T^{**}.

In Section 5.3 we take an operator ideal approach to the study of sequentially limited operators.

Let $(\mathcal{A}, \alpha)$ be a Banach ideal of operators. With the vector space $\mathcal{A}(X, Y)$ we associate

$$\mathcal{A}_\Lambda(X, Y) := \{T \in \mathcal{L}(X, Y) : ST \in \mathcal{A}(X, \Lambda),\ \forall S \in \mathcal{L}(Y, \Lambda)\}.$$

From the operator ideal properties of $\mathcal{A}$ we verify that $\mathcal{A}_\Lambda$ also defines an operator ideal. We now define

$$\alpha_\Lambda(T) := \sup\{\alpha(ST) : S \in \mathcal{L}(Y, \Lambda), \|S\| \leq 1\}$$

and verify that $\alpha_\Lambda(\cdot)$ defines a norm on $\mathcal{A}_\Lambda(X, Y)$. We then show that $(\mathcal{A}_\Lambda, \alpha_\Lambda(\cdot))$ is a Banach operator ideal.

Recall that an operator $T \in \mathcal{L}(X, Y)$ belongs to the component $\mathcal{A}^{max}(X, Y)$ of the *maximal hull* $\mathcal{A}^{max}$ of an ideal $\mathcal{A}$ if $RTS \in \mathcal{A}(X_0, Y_0)$ for all $S \in \bar{\mathcal{F}}(X_0, X), \forall R \in \bar{\mathcal{F}}(Y, Y_0)$ and for all Banach spaces X_0, Y_0. The ideal $\mathcal{A}$ is called *maximal* if $\mathcal{A} = \mathcal{A}^{max}$, i.e if $\mathcal{A}(X, Y) = \mathcal{A}^{max}(X, Y)$ for all Banach spaces X, Y. A Banach operator ideal $(\mathcal{A}, \alpha)$ is a maximal Banach ideal if $(\mathcal{A}, \alpha) = (\mathcal{A}^{max}, \alpha^{max})$ (isometrically).

Using results about tensor norms and the fact that a maximal Banach operator ideal is associated with a finitely generated tensor norm, it follows that:

If $(\mathcal{A}, \alpha)$ is a maximal Banach operator ideal, then $T \in \mathcal{A}(X, Y)$ if and only if $T^{**} \in \mathcal{A}(X^{**}, Y^{**})$ for all Banach spaces X, Y. In this case $\alpha(T) = \alpha(T^{**})$.

From this we obtain:

Let Λ be a reflexive BK-space with AK and suppose $(\mathcal{A}, \alpha)$ is a maximal Banach operator ideal. Then $T \in \mathcal{A}_\Lambda(X, Y)$ if and only if $T^{**} \in$

$\mathcal{A}_\Lambda(X^{**}, Y^{**})$. In this case $\alpha_\Lambda^{max}(T) = \alpha_\Lambda^{max}(T^{**})$ and $\alpha_\Lambda(T) \le \alpha_\Lambda(T^{**})$.

The maximality of the Banach ideal (Π_p, π_p) and previous result now implies that a much stronger version of our proposition in section 5.3 is true:

Let $1 < p < \infty$. A bounded linear operator $T : X \to Y$ is sequentially p-limited if and only if $T^{**} : X^{**} \to Y^{**}$ is sequentially p-limited.

In Section 5.4 we consider some more classes of operators.

An operator $T : X \to Y$ is called (q,p)-summing (with $1 \le p, q < \infty$) if there is an induced operator

$$\widehat{T} : \ell_p^{weak}(X) \to \ell_q^{strong}(Y) : (x_n) \mapsto (Tx_n).$$

The vector space of (q,p)-summing operators is denoted by $\Pi_{q,p}(X,Y)$; it is normed by the norm

$$\pi_{q,p}(T) = \|\widehat{T}\|,$$

where $\|\widehat{T}\|$ denotes the operator norm of $\widehat{T}$.

Let $1 \le p \le q < \infty$ and let $1 \le r < \infty$. Let $(\mathcal{A}, \alpha) = (\Pi_{q,p}, \pi_{q,p})$, then:

1. We denote $(\mathcal{A}_{\ell_r}, \alpha_{\ell_r})$ by $(Lt_{q,p,r}, \ell t_{q,p,r})$. In this case we have $T \in Lt_{q,p,r}(X,Y)$ if and only if $ST \in \Pi_{q,p}(X, \ell_r)$ for all $S \in \mathcal{L}(Y, \ell_r)$, i.e. if and only if

   $$\sum_{n=1}^{\infty} \|STx_n\|_r^q < \infty, \quad \forall\, S \in \mathcal{L}(Y, \ell_r), \forall\, (x_n) \in \ell_p^{weak}(X).$$

   Also, for $T \in Lt_{q,p,r}(X,Y)$, we have

   $$\ell t_{q,p,r}(T) = \sup_{S \in U_{\mathcal{L}(Y,\ell_r)}} \pi_{q,p}(ST).$$

2. In case of $p = q = r$, we clearly have $(Lt_{p,p,p}, \ell t_{p,p,p}) = (Lt_p, \ell t_p)$.

3. In case of $p = q$, we denote $(Lt_{q,p,r}, \ell t_{q,p,r})$ by $(Lt_{p,r}, \ell t_{p,r})$. In this case we have $T \in Lt_{p,r}(X,Y)$ if and only if $ST \in \Pi_p(X, \ell_r)$ for all $S \in \mathcal{L}(Y, \ell_r)$. The operators $T \in Lt_{p,r}(X,Y)$ will be called sequentially (p,r)-limited. In case of $p = r$, we again have $(Lt_{p,p}, \ell t_{p,p}) = (Lt_p, \ell t_p)$

Then the pairs $(Lt_{q,p,r}, \ell t_{q,p,r})$ and $(Lt_{p,r}, \ell t_{p,r})$ are Banach operator ideals. We show that if $1 \leq p \leq q < \infty$ and $1 \leq r < \infty$, then $Lt_{p,r}(X,Y) \subseteq Lt_{q,r}(X,Y)$. Moreover, for $T \in Lt_{p,r}(X,Y)$ we have $\ell t_{q,r}(T) \leq \ell t_{p,r}(T)$.

Let $1 \leq q < \infty$. Recall that $T \in \mathcal{L}(X,Y)$ is said to be q-nuclear if and only if it has a representation

$$T = \sum_{i=1}^{\infty} x_i^* \otimes y_i,$$

where $(x_i^*) \in \ell_q^{strong}(X^*)$ and $(y_i) \in \ell_{q'}^{weak}(Y)$. The norm on the space $\mathcal{N}_q(X,Y)$ of q-nuclear operators is then given by

$$\nu_q(T) := \inf\{\|(x_i^*)\|_q^{strong}\|(y_i)\|_{q'}^{weak} : T = \sum_{i=1}^{\infty} x_i^* \otimes y_i\}.$$

It follows that if $\mathcal{A} = \mathcal{N}_q$ (and $\alpha = \nu_q$) and $\Lambda = \ell_r$, then $\mathcal{A}_\Lambda$ becomes $(\mathcal{N}_q)_r$ and $\alpha_\Lambda = (\nu_q)_r$. We then have the following composition result:

Let $1 \leq p, q, r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $T \in \mathcal{N}_q(X,Y)$ and $S \in Lt_{p,r}(Y,Z)$. Then $ST \in (\mathcal{N}_r)_r(X,Z)$ and

$$(\nu_r)_r(ST) \leq \ell t_{p,r}(S)\, \nu_q(T).$$

In Chapter 6 we study Dunford-Pettis-type functions on Banach spaces and Banach lattices.

In Section 6.1 we review a few well-known definitions and results regarding polynomials in general and n-homogeneous polynomials in particular. We recall that if X and Y are vector spaces over $\mathbb{C}$, then a mapping $P : X \to Y$ is called an n-homogeneous polynomial from X to Y, if there exists an element $L \in \mathcal{L}_a(^nX;Y)$ such that $P = L \circ \Delta_n$; that is we have $P(x) = L(x, x, \ldots, x)$ for all $x \in X$. The following facts about n-homogeneous polynomials are known in the literature:

Let X be a locally convex space, Y a normed linear space and suppose $P \in \mathcal{P}_a(^nX;Y)$. The following are equivalent:

(a) P is locally uniformly continuous (i.e. for each $x \in X$ there exists a neighbourhood V of x such that $P|_V$ is uniformly continuous).

(b) P is everywhere continuous.

(c) P is continuous at some point.

(d) P is bounded on a neighbourhood of some point in X.

(e) P is a locally bounded function (i.e. every point in X contains a neighbourhood on which P is bounded).

When a Banach space X has the Dunford-Pettis property, then we have the following interesting results:

(a) If $P \in \mathcal{P}(^n X)$, then P is weakly sequentially continuous.

(b) If ℓ_1 is not isomorphic to a subspace of X, then continuous polynomials on X are (uniformly) weakly continuous on bounded sets.

In Section 6.2 we take a quick tour of the holomorphic mappings. Let U be an open subset of X. A mapping $f : U \to Y$ is said to be holomorphic (or analytic) if for each $a \in U$ there exists a ball $B(a, r) \subset U$ and a sequence of polynomials $P_m \in \mathcal{P}(^n X; Y)$ such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - a),$$

uniformly on $B(a, r)$. It should be noted that each polynomial (as a linear combination of m-homogeneous polynomials) is holomorphic.

In Section 6.3 we study the so-called Dunford-Pettis-type operators. In [37] we introduced the p-convergent functions on Banach spaces. In this section we extend this notion to the so-called disjoint p-convergent functions from a Banach lattice to a Banach space. We introduce and characterise the notions of disjoint Dunford-Pettis and Dunford-Pettis* properties of order p (abbreviated as disjoint DPP_p and disjoint DP^*P_p respectively) on a Banach lattice. We show that if we let $1 \leq p < \infty$, E be a Banach lattice and X be a Banach space and assume that X contains an isomorphic copy of c_0, then if every $T \in \mathcal{L}(E, X)$ is disjoint p-convergent, then E has the disjoint DP^*P_p. In this case, every polynomial $P \in \mathcal{P}(^n E, X)$ is a disjoint p-convergent function for all $n \in \mathbb{N}$.

It is well-known that if W is a weakly relatively compact subset of a Banach lattice, then every disjoint sequence in the solid hull of W converges weakly to zero. In particular, this implies that all disjoint weak p-convergent sequences in a Banach lattice E have limit 0 and so they belong to $\ell_p^{weak}(E)$. We call a subset A of a Banach lattice E "disjoint weakly p-compact" if it is weakly p-compact and its elements are mutually disjoint. We may then characterise the disjoint DP^*P_p as follows:

A Banach lattice E has the disjoint DP^*P_p if and only if all disjoint weakly p-compact sets in E are limited.

If a sequence of k-homogeneous polynomials $(P_n) \subset \mathcal{P}(^kX)$ converges pointwise to a limit P (also a k-homogeneous polynomial), then $P_n \to P$ uniformly on all limited subsets of X. Using this fact, we have the following result in connection with pointwise convergence of k-homogeneous polynomials on Banach lattices with the disjoint DP^*P_p:

Let E be a Banach lattice with the disjoint DP^*P_p and let $(P_n) \subset \mathcal{P}(^kE)$ such that $P_n \to P \in \mathcal{P}(^kE)$ pointwise. Then $P_n \to P$ uniformly on all disjoint weakly p-compact sets in E and for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$, we have $P_n(x_n) \xrightarrow[\infty]{n} 0$.

Recall that in a Gelfand-Phillips space limited sets are relatively (norm) compact. For polynomials and holomorphic functions on Banach lattices with values in a Banach space with the Gelfand-Phillips property we have that if E has the disjoint DP^*P_p and X is a Gelfand-Phillips space, then every $P \in \mathcal{P}(^nE, X)$ is disjoint p-convergent. Furthermore, each $f \in \mathcal{H}(E, X)$ which is bounded on limited sets, is weakly continuous on disjoint weakly p-compact sets.

The following two interesting facts on coordinate pairs aid in another characterisation of the disjoint DP^*P_p:

Let E, F be Banach lattices and $(x_n) \in E^{\mathbb{N}}$, $(y_n) \in F^{\mathbb{N}}$. Then:

(i) (x_n, y_n) is a disjoint sequence in $E \times F$ if and only if (x_n) is a disjoint sequence in E and (y_n) is a disjoint sequence in F.

(ii) $(x_n, y_n) \in \ell_p^{weak}(E \times F)$ if and only if $(x_n) \in \ell_p^{weak}(E)$ and $(y_n) \in$

$\ell_p^{weak}(F)$.

Let X, Y and Z be Banach spaces. A bilinear operator

$$\phi : X \times Y \to Z$$

is called separately compact if for each fixed $y \in Y$, the linear operator $T_y : X \to Z : \ x \mapsto \phi(x,y)$ is compact and for each fixed $x \in X$, the linear operator $T_x : Y \to Z : \ y \mapsto \phi(x,y)$ is compact. Now, if E is a Banach lattice, then if every symmetric bilinear separately compact map $E \times E \to c_0$ is disjoint p-convergent, then E has the disjoint DP^*P_p.

If E is a Banach lattice and X is a Banach space, then if $f \in \mathcal{H}(E,X)$ is disjoint p-convergent, we show that it is bounded on all disjoint weakly p-compact subsets of E.

Recall that a subset L of X is called bounding if every $f \in \mathcal{H}(X)$ is bounded on L. We denote the class of all Banach lattices whose disjoint weakly p-compact subsets are bounding by $\mathbb{B}_p$. Then if E is a Banach lattice, and if each $f \in \mathcal{H}(E)$ is disjoint p-convergent, then $E \in \mathbb{B}_p$. On the other hand, if $E \in \mathbb{B}_p$, then each $f \in \mathcal{H}(E)$ is weakly continuous on disjoint weakly p-compact sets. Furthermore, if E is a Banach lattice and X a Banach space, with $E \in \mathbb{B}_p$ and X a Gelfand-Phillips space, then each $f \in \mathcal{H}(E,X)$ is disjoint p-convergent. Finally, if E is a Banach lattice, then each $f \in \mathcal{H}(E)$ is disjoint p-convergent if and only if $E \in \mathbb{B}_p$.

Chapter 1

Preliminaries

1.1 Basic information on Banach spaces

Throughout the text we use X, Y, Z etc. to denote Banach spaces (over $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$). We denote by $\mathcal{L}(X,Y)$ (respectively, $\mathcal{K}(X,Y)$ and $\mathcal{W}(X,Y)$) the space of bounded (respectively, compact and weakly compact) linear operators from X to Y and the identity operator on X is denoted by id_X. The continuous dual space $\mathcal{L}(X,\mathbb{K})$ is denoted by X^* and the closed unit ball of X by $\mathcal{B}_X$. The weak topology on X is denoted by $\sigma(X,X^*)$ and $\sigma(X^*,X)$ denotes the weak* topology on X^*. As is custom, we agree to use E, F, G etc. to denote Banach lattices. In this thesis we will throughout assume that the Banach lattices are real, i.e. they are linear spaces over $\mathbb{R}$.

Let $1 \leq p < \infty$. The conjugate number will be denoted by p', i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. The Banach space of p-summable scalar sequences (for $1 \leq p < \infty$) is denoted by ℓ_p and ℓ_∞ is the space of bounded scalar sequences. The closed subspace of ℓ_∞ consisting of the scalar sequences which are convergent (respectively, convergent with limit 0) in the norm of ℓ_∞, is denoted by c (respectively, c_0). The unit coordinate vector e_n in these sequence spaces, is the sequence $e_n = (\delta_{n,j})_j$, where $\delta_{n,j} = 0$ if $j \neq n$ and $\delta_{n,n} = 1$.

We denote the space of all weakly p-summable sequences in a Banach space X by $\ell_p^{weak}(X)$ and recall that it is a Banach space with norm

$$\|(x_i)\|_p^{weak} := \sup\left\{\left(\sum_{i=1}^{\infty} |\langle x_i, x^*\rangle|^p\right)^{1/p} : x^* \in X^*, \|x^*\| \leq 1\right\}.$$

This space is isometrically isomorphic to $\mathcal{L}(\ell_{p'}, X)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$). For $(x_i) \in \ell_p^{weak}(X)$ (taking $1 \leq p < \infty$), the linear operator

$$E_{(x_i)} : \ell_{p'} \to X : (\lambda_i) \mapsto \sum_{i=1}^{\infty} \lambda_i x_i$$

is bounded, with $\|E_{(x_i)}\| = \|(x_i)\|_p^{weak}$. Conversely, it is also well-known that each $T \in \mathcal{L}(\ell_{p'}, X)$ can be uniquely identified as an operator $E_{(x_i)}$ for some $(x_i) \in \ell_p^{weak}(X)$, so that $\mathcal{L}(\ell_{p'}, X)$ is isometrically identified with $\ell_p^{weak}(X)$ by the mapping $(x_i) \mapsto E_{(x_i)}$. In the case of $p = \infty$ we consider the space $c_0^{weak}(X)$ of weak null sequences in X.

The space of all weak* p-summable sequences in the dual space X^* of a Banach space X is denoted by $\ell_p^{weak^*}(X^*)$. Recall that it is a Banach space with norm

$$\|(x_i^*)\|_p^{weak^*} := \sup\left\{\left(\sum_{i=1}^{\infty} |\langle x, x_i^*\rangle|^p\right)^{1/p} : x \in X, \|x\| \leq 1\right\}.$$

This space is isometrically isomorphic to $\mathcal{L}(X, \ell_p)$.
For a fixed $(x_i^*) \in \ell_p^{weak^*}(X^*)$ the operator $F_{(x_i^*)} : X \to \ell_p : x \mapsto (\langle x, x_i^*\rangle)_i$ is bounded and linear with $\|F_{(x_i^*)}\| = \|(x_i^*)\|_p^{weak^*}$. Conversely, since each $T \in \mathcal{L}(X, \ell_p)$ can be uniquely identified with an operator $F_{(x_i^*)}$ for some $(x_i^*) \in \ell_p^{weak^*}(X^*)$, the mapping $(x_i^*) \mapsto F_{(x_i^*)}$ identifies $\mathcal{L}(X, \ell_p)$ isometrically with $\ell_p^{weak}(X^*)$. In the case of $p = \infty$ we consider the space $c_0^{weak^*}(X^*)$ of weak* null sequences in X^*. Note that $\ell_p^{weak^*}(X^*) = \ell_p^{weak}(X^*)$, but that $c_0^{weak}(X^*) \subseteq c_0^{weak^*}(X^*)$, whereby $c_0^{weak}(X^*)$ is isometrically isomorphic to the space $\mathcal{W}(X, c_0)$ of weakly compact operators.

In general it is not true that $\lim_{n\to\infty} \|(x_i) - (x_1, x_2, \ldots x_n, 0, 0\ldots)\|_p^{weak} = 0$. The subspace $\ell_p^u(X)$ of $\ell_p^{weak}(X)$ for which this is true, is a Banach space with respect to the norm $\|(\cdot)\|_p^{weak}$ and the identification of $(x_i) \in \ell_p^u(X)$ with $E_{(x_i)} : \ell_{p'} \to X : (\lambda_i) \mapsto \sum_{i=1}^{\infty} \lambda_i x_i$ defines an isometric isomorphism between $\ell_p^u(X)$ and the space $K(\ell_{p'}, X)$ of compact linear operators. Refer (for instance) to the paper [35] and [25] (Chapter 1, section 8) for these facts.

In the case of Banach lattices we have the following "solidness" result for positive sequences:

Remark 1.1.1 *Suppose E is a Banach lattice and $(x_n) \in \ell_p^{weak}(E)$ satisfies $x_n \geq 0$ for all n. Suppose $y_n \in E$ satisfies $0 \leq y_n \leq x_n$ for all n. Given $x^* \in E^*$, then we have $(\langle (x^*)^+, x_n\rangle) \in \ell_p$ and $(\langle (x^*)^-, x_n\rangle) \in \ell_p$. Also $\langle (x^*)^+, y_n\rangle \leq \langle (x^*)^+, x_n\rangle$ and $\langle (x^*)^-, y_n\rangle \leq \langle (x^*)^-, x_n\rangle$ for all n. Thus $(\langle x^*, y_n\rangle) \in \ell_p$. This shows that $(y_n) \in \ell_p^{weak}(E)$ as well.*

1.2 Basic information on Banach lattices

Most of the work in this section is scattered amongst various texts; see for instance [3], [50] and [59]. I would like to thank Prof. J. J. Grobler for availing his class notes on vector lattices, which incidentally provided me with a good summary of most of the work covered in the referenced texts.

Definition 1.2.1 *A set X on which a transitive, reflexive and anti-symmetric binary relation $\leq$ is defined, is called a* partially ordered set.

This means that in a partially ordered set $(X, \leq)$ the relation $\leq$ satisfies the following three conditions:

1. Transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$;
2. Reflexivity: $x \leq x$ for all $x \in X$;
3. Anti-symmetry: $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in X$.

We also write $y \geq x$ if $x \leq y$, and $x < y$ if $x \leq y$ and $x \neq y$. If $x \leq y$ or $y \leq x$ we say that x and y are *comparable.*

A subset $Y \subset X$ is called *bounded from above* if there is an element $x \in X$ such that $y \leq x$ for all $y \in Y$. Such an element x is called an *upper bound* for Y. An upper bound x_0 for Y with the property that $x_0 \leq x$ for every upper bound x of Y, is called the *least upper bound* or *supremum* of Y and is denoted by $\sup Y$.

The notions of *bounded from below, lower bound* and *greatest lower bound* or *infimum* for a set Y are defined similarly. The infimum of Y is denoted by $\inf Y$.

The supremum and infimum of a set are unique if they exist.

Definition 1.2.2 *A real vector space E which is partially ordered by the relation $\leq$ is called an* ordered vector space *if*

(i) $x \leq y$ *implies that* $x + z \leq y + z$ *for all* $x, y, z \in E$;

(ii) $x \leq y$ *implies that* $\lambda x \leq \lambda y$ *for all* $x, y \in E$ *and* $0 \leq \lambda \in \mathbb{R}$.

The ordered vector space is called a Riesz space or vector lattice if $x \vee y$ and $x \wedge y$ exist for all $x, y \in E$.

The subset $E^+ := \{x \in E : x \geq 0\}$ is called the *positive cone* of the ordered vector space E. The elements of E^+ are called the *positive elements* of E. The positive cone has the following properties which are easily derived from the definition:

1. $x, y \in E^+$ implies that $x + y \in E^+$;
2. $x \in E^+$ and $0 \leq \lambda \in \mathbb{R}$ implies that $\lambda x \in E^+$;
3. $x \in E^+$ and $-x \in E^+$ implies that $x = 0$.

Conversely, a set E^+ in a linear space E is called a *proper cone* if it has properties 1-3 above. If we define in E the relation $x \leq y$ to mean that $y - x \in E^+$, then $(E, \leq)$ is a partially ordered vector space.

Definition 1.2.3 *Let E be a Riesz space. For every $x \in E$ we define*

$$x^+ := x \vee 0; \quad x^- := (-x) \vee 0; \quad |x| := x \vee (-x).$$

to be the positive part, *the* negative part *and the* absolute value *of x respectively.*

It is immediately clear that x^+ and x^- are in E^+ and that $|-x| = |x|$. Also, $(-x)^- = x^+$ and $(-x)^+ = x^-$.

The following are the elementary properties of these elements. The proofs may be found in [3]:

Theorem 1.2.4 *Let E be a Riesz space. Then the following hold for every $x \in E$.*

(i) $x = x^+ - x^-$, $\quad x^+ \wedge x^- = 0$, *and* $\quad |x| = x^+ + x^- \in E^+$.

(ii) $0 \leq x^+ \leq |x|$ *and* $0 \leq x^- \leq |x|$.

(iii) $-x^- \leq x \leq x^+$.

(iv) $x \leq y$ *if and only if* $x^+ \leq y^+$ *and* $x^- \geq y^-$.

The proof of the next theorem is trivial:

Theorem 1.2.5 *If E is a Riesz space then*

(i) $x \vee y = (x-y)^+ + y = (y-x)^+ + x$;

(ii) $x \wedge y = x - (x-y)^+ = y - (y-x)^+$;

(iii) $x \vee y - x \wedge y = |x-y|$;

(iv) $x \vee y + x \wedge y = x + y$.

Definition 1.2.6 *The elements x and y in the Riesz space E are called* disjoint *whenever $|x| \wedge |y| = 0$.*

If x and y are disjoint we write $x \perp y$. Two subsets A and B in E are called disjoint whenever $a \perp b$ for every $a \in A$ and $b \in B$.

If $A \subset E$, we define the disjoint complement *of A as the set $A^d := \{x \in E : x \perp a$ for every $a \in A\}$.*

A subset $S \subset E$ is called a disjoint system *whenever $0 \notin S$ and $x \perp y$ for every $x, y \in S$.*

We recall that a *directed set* Γ is a partially ordered set with the property that if $\alpha,\ \beta \in \Gamma$, there exists an element $\gamma \in \Gamma$, such that γ is an upper bound for the set $\{\alpha, \beta\}$, i.e., an element γ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$.

A *net* $(x_\alpha)_{\alpha \in \Gamma}$ in E is a function defined on Γ with values in E.

Definition 1.2.7 A net $(x_\alpha)_{\alpha\in\Gamma}$ in E is called *increasing* whenever $x_\alpha \leq x_\beta$ if $\alpha \leq \beta$ and it is called *decreasing* whenever $x_\alpha \geq x_\beta$ if $\alpha \leq \beta$.

If $(x_\alpha)_{\alpha\in\Gamma}$ is increasing, we write $x_\alpha \uparrow$ and if, moreover, $x = \sup x_\alpha$ we write $x_\alpha \uparrow x$.

Similarly, if the net $(x_\alpha)_{\alpha\in\Gamma}$ is decreasing, we write $x_\alpha \downarrow$ and if, moreover, $x = \inf x_\alpha$ we write $x_\alpha \downarrow x$.

Definition 1.2.8 A net $(x_\alpha)_{\alpha\in\Gamma}$ is called *order convergent* to x if there exists a net $(y_\alpha)_{\alpha\in\Gamma}$ satisfying $y_\alpha \downarrow 0$ and $|x - x_\alpha| \leq y_\alpha$ for all $\alpha \in \Gamma$. We write $x = \text{o-lim}_{\alpha\in\Gamma} x_\alpha$ or simply $x_\alpha \to x$ in order.

A sequence (x_n) is called *order convergent* to x as $n \to \infty$ if there exists a decreasing sequence $y_n \downarrow 0$ such that $|x - x_n| \leq y_n$ for all $n \in \mathbb{N}$. We write $x = \text{o-lim}_{n\to\infty} x_n$ or $x_n \to x$ in order as $n \to \infty$.

Definition 1.2.9 A Riesz space E is called *Archimedean* if, for all $x, y \in E$, it follows from $nx \leq y$ for all $n \in \mathbb{N}$ that $x \leq 0$.

Definition 1.2.10

1. A linear subspace $G \subset E$ is called a *Riesz subspace* or a *sublattice* of the Riesz space E if $x \vee y$ and $x \wedge y$ belongs to G for all $x, y \in G$.
2. A subset A is called *solid* if $|x| \leq |y|,\ y \in A \Longrightarrow x \in A$.
3. A solid linear subspace of the Riesz space E is called an *ideal.*

Definition 1.2.11 The Riesz space E is said to be *Dedekind complete* or *order complete* whenever every non-empty subset which is bounded above has a least upper bound.

Definition 1.2.12 Let E and F be Riesz spaces and let $T : E \to F$ be a linear operator. Then

1. T is called *positive* (denoted by $T \geq 0$) whenever $Tx \geq 0$ for all $x \geq 0$;
2. T is called a *Riesz homomorphism* or *lattice homomorphism* whenever $T(x \vee y) = Tx \vee Ty$.
3. T is called *order continuous* whenever $Tx_\alpha \to 0$ in order for every net (x_α) satisfying $x_\alpha \to 0$ in order.

If T is positive, then it follows from $|x| \geq \pm x$ for all $x \in E$, that $T|x| \geq \pm Tx$ for all $x \in E$. Hence, $T|x| \geq Tx \vee (-Tx) = |Tx|$.

Conversely, if $T|x| \geq |Tx|$ for all $x \in E$, then, for $x \in E^+$ we have $Tx = T|x| \geq |Tx| \geq 0$. It follows that T is positive.

We thus obtain:

Proposition 1.2.13 *The linear operator $T : E \to F$ is positive if and only if $|Tx| \leq T|x|$ for all $x \in E$.*

We note that every Riesz homomorphism is positive; in fact, if $x \in E^+$, then

$$Tx = Tx^+ = T(x \vee 0) = Tx \vee T0 = Tx \vee 0 = (Tx)^+ \geq 0.$$

Theorem 1.2.14 *Let $T : E \to F$ be a linear operator from the Riesz space E into the Riesz space F. Then the following are equivalent.*

1. *T is a Riesz homomorphism.*
2. *$T(x \wedge y) = Tx \wedge Ty$ for all $x, y \in E$.*
3. *$Tx^+ \wedge Tx^- = 0$ for all $x \in E$.*
4. *$Tx^+ = (Tx)^+$ and $Tx^- = (Tx)^-$ for all $x \in E$.*
5. *$|Tx| = T|x|$ for all $x \in E$.*
6. *$Tx^+ = (Tx)^+$ for all $x \in E$.*

Proposition 1.2.15 *If T is a Riesz homomorphism of E onto an ideal in F, then $T[A]$ is a solid subset of F for every solid subset $A \subset E$.*

Definition 1.2.16 A bijective Riesz homomorphism of a Riesz space E onto a Riesz space F is called a *Riesz isomorphism* or a *lattice isomorphism.*

It is immediately clear that if T is a Riesz isomorphism of E onto F then T^{-1} is also a Riesz isomorphism. Every Riesz isomorphism T is positive, onto and one-one. The converse is not true.

Definition 1.2.17 A linear operator $T : E \to F$ is called *order bounded* if it maps order bounded subsets into order bounded subsets.

Let E be a Riesz space. A seminorm p on E is called a *Riesz semi-norm* or also a *lattice semi-norm* if it follows from $|x| \leq |y|$ in E that $p(x) \leq p(y)$ for all $x, y \in E$. If the lattice semi-norm p is a norm, then the pair (E, p) is called a *normed Riesz space,* or also a lattice normed vector lattice. If the space E is complete with reference to the lattice norm p, then (E, p) is called a *Banach lattice.*

The semi-norm p is a lattice semi-norm if and only if the following two conditions hold

1. $p(x) = p(|x|)$ for all $x \in E$,
2. $p(u) \leq p(v)$ for all $0 \leq u \leq v$ in E.

It is also easy to see that a normed Riesz space is necessarily Archimedean and that the unit sphere in a normed Riesz space is solid.

We note the following simple properties of normed Riesz spaces.

Theorem 1.2.18 *Let E be a lattice normed vector lattice. The following hold:*

(a) *The maps $x \mapsto x^+$, $x \mapsto x^-$, $x \mapsto |x|$ and $(x, y) \mapsto x \vee y$, $(x, y) \mapsto x \wedge y$, are uniformly continuous from E (respectively $E \times E$) into E.*

(b) *E^+ is a closed subset of E.*

(c) *The closure of a solid subset is solid. In particular, the closure of an ideal in E is again an ideal in E.*

(d) *If $x_\tau \uparrow$ and $\|x - x_\tau\| \to 0$, then $x = \sup x_\tau$. In particular, if $x_n \uparrow$ and if $\|x - x_n\| \to 0$, then $x = \sup x_n$.*

(e) *Every band in E is closed.*

(f) *Every band projection is continuous.*

Theorem 1.2.19 *Let E be a Banach lattice and let F be a normed Riesz space. Then every positive linear operator from E into F is continuous.*

From this we may obtain:

Corollary 1.2.20 *Let E be a Banach lattice and F a normed Riesz space. Then the following statements hold.*

(a) *If S and T are linear operators of E into F satisfying $|Su| \leq Tu$ for all $u \in E^+$, then S and T are continuous.*

(b) *Every positive linear functional on E is continuous.*

(c) *Every maximal ideal in E is closed.*

In many examples of normed Riesz spaces, order convergence implies norm convergence. In these spaces the connection between order and norm is more intimate and we can expect them to be well-behaved. Examples of such spaces are the spaces $L^p(X, \Sigma, \mu)$ with $1 \leq p < \infty$, ℓ^p with $1 \leq p < \infty$ and c_0. The spaces $L^\infty(X, \Sigma, \mu)$ and ℓ^∞ do not have this property except in the finite-dimensional case.

Definition 1.2.21 A normed Riesz space E is said to have an *order continuous norm* whenever $\|x_\alpha\| \downarrow 0$ for every downwards directed net (x_α) satisfying $x_\alpha \downarrow 0$.

We now formulate one of the main theorems on order continuity of the norm.

Theorem 1.2.22 *Let E be a Banach lattice. The following statements are equivalent.*

(i) *E is Dedekind complete and every continuous linear functional is order continuous.*

(ii) *Every upwards directed order bounded net in E is weakly convergent.*

(iii) *If the net (x_α) satisfies $x_\alpha \downarrow 0$, then $\|x_\alpha\| \downarrow 0$.*

(iv) *E is σ-Dedekind complete and if the sequence $x_n \downarrow 0$, then $\|x_n\| \downarrow 0$.*

(v) *The image of E in E^{**} is an ideal.*

(vi) *Every order interval in E is weakly compact.*

The theory of AM-spaces is of particular importance in the theory of Banach lattices.

Definition 1.2.23 The norm on a Riesz space E is said to be an *M-norm* if

$$\|x \vee y\| = \sup\{\|x\|, \|y\|\} \text{ for all } x, y \in E^+.$$

The space E furnished with an M-norm is called an *M-normed space.* An M-normed Banach lattice is called an *abstract M-space* or also an AM-space.

If the unit ball U of an AM-space has a largest element e, we call e the *unit* of the AM-space. In this case $U = [-e, e]$.

It follows from the definition above that $x \in U$ if and only if $|x| \leq e$.

Theorem 1.2.24 *Let E be an Archimedean Riesz space with order unit e. The gauge function p_e of the interval $[-e, e]$, which is defined as*

$$p_e(x) := \inf\{l > 0 \, : \, |x| \leq le\} = \inf\{l > 0 : x \in l[-e, e]\}$$

is an M-norm on E.

Definition 1.2.25 The Riesz space E is called *uniformly complete* whenever $\sup\{\sum_{k=1}^{n} x_k : n \in \mathbb{N}\}$ exists in E for every uniformly ℓ_1-bounded sequence $(x_n) \subset E^+$.

Theorem 1.2.26 *Let E be an Archimedean Riesz space with order unit e. Then (E, p_e) is an AM-space with unit e if and only if E is uniformly complete.*

Corollary 1.2.27 *If E is a Banach lattice and if $x \in E^+$, then the principal ideal E_x, generated in E by x, is an AM-space with norm the gauge function of the interval $[-x, x]$ and with unit x. The embedding $E_x \to E$ is continuous.*

The most important examples of AM-spaces with unit are the spaces $C(K)$, with K a compact Hausdorff space.

Abstract L-spaces (AL-spaces) are spaces which have the properties of the spaces $L^1(X, \Sigma, \mu)$. In certain respects these spaces and AM-spaces find themselves at opposite ends of a spectrum of spaces. Their properties are very different, but at the same time there is an intimate connection between them.

Definition 1.2.28 A Riesz norm on a Riesz space E is called an *L-norm* if it satisfies the condition that

$$\|x+y\| = \|x\| + \|y\| \text{ for all } x, y \in E^+.$$

An L-normed Riesz space is called an *abstract L-space* (or an *AL-space*) if it is norm complete.

An AL-space is therefore a Banach lattice, the norm of which is additive on the positive cone.

Theorem 1.2.29 *An AL-space E is Dedekind complete and has order-continuous norm (cf. [3]).*

Chapter 2

Classes of Dunford-Pettis operators

In this chapter we provide a review of existing results on Dunford-Pettis, weak Dunford-Pettis, almost Dunford-Pettis and weak* Dunford-Pettis operators, and their applications in the study of geometrical properties of Banach spaces and Banach lattices.

2.1 Dunford-Pettis operators

In their 1940 landmark paper, "Linear operations on summable functions", N. Dunford and P.J. Pettis proved amongst other things, that a weakly compact operator $T : L_1(\mu) \to L_1(\mu)$ carries weakly convergent sequences to norm convergent sequences. It was Grothendieck who in his 1953 paper, "Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$" called every operator with this property a Dunford-Pettis operator. Aliprantis and Burkinshaw in [3] define these operators as follows:

Definition 2.1.1 *A bounded linear operator $T : X \to Y$ between two Banach spaces, is said to be a Dunford-Pettis operator whenever $(x_n) \in c_0^{weak}(X)$ implies $\|Tx_n\| \to 0$. Dunford-Pettis operators are also called* **completely continuous operators**.

Remark 2.1.2 *The following are well-known facts with regards to Dunford-Pettis operators (cf. [3]):*

(i) Every Dunford-Pettis operator is continuous.

(ii) A compact operator is necessarily a Dunford-Pettis operator.

(iii) If X is a reflexive Banach space, then an operator with domain X is Dunford-Pettis if and only if it is compact.

(iv) A Dunford-Pettis operator need not be a compact operator.

Furthermore, Theorem 5.79 in [3], page 340 characterises the Dunford-Pettis operators as follows in terms of Cauchy sequences:

Theorem 2.1.3 *A bounded linear operator $T : X \to Y$ between two Banach spaces, is a Dunford-Pettis operator if and only if T carries weakly Cauchy sequences of X to norm convergent sequences of Y.*

This result along with Rosenthal's ℓ_1-theorem immediately yields:

Theorem 2.1.4 *If ℓ_1 does not embed in a Banach space X, then every Dunford-Pettis operator from X to an arbitrary Banach space is compact.*

We characterise the Dunford-Pettis operators in terms of compact and weakly compact operators as follows:

Theorem 2.1.5 *(cf. [3], page 341) For a bounded linear operator $T : X \to Y$ between two Banach spaces, the following statements are equivalent:*

(1) T is a Dunford-Pettis operator.

(2) T carries weakly relatively compact subsets of X to norm totally bounded subsets of Y.

(3) For an arbitrary Banach space Z and every weakly compact operator $S : Z \to X$, the operator TS is a compact operator.

(4) For every weakly compact operator $S : \ell_1 \to X$, the operator TS is compact.

Recall that a Banach space X has the Dunford-Pettis property whenever $(x_n) \in c_0^{weak}(X)$ and $(x_n^*) \in c_0^{weak}(X^*)$ imply $\lim x_n^*(x_n) = 0$ or equivalently, if we let

$$x_n^*(x_n) = x_n^*(x) + x^*(x_n - x) + (x_n^* - x^*)(x_n - x),$$

then it is clear that a Banach space X has the Dunford-Pettis property if and only if $x_n \xrightarrow{weak} x$ in X and $x_n^* \xrightarrow{weak} x^*$ in X^* imply $x_n^*(x_n) \to x^*(x)$. Furthermore, the Dunford-Pettis property is characterised in terms of the Dunford-Pettis operators as follows:

Theorem 2.1.6 *([3], Theorem 5.82, page 342) For a Banach space X the following are equivalent:*

(1) X has the Dunford-Pettis property.

(2) Every weakly compact operator from X to an arbitrary Banach space maps weakly compact sets to norm compact sets.

(3) Every weakly compact operator from X to an arbitrary Banach space is a Dunford-Pettis operator.

(4) Every weakly compact operator from X to c_0 is a Dunford-Pettis operator.

Aliprantis and Burkinshaw, amongst others, studied the question: if a positive operator S between Banach lattices is dominated by a Dunford-Pettis operator, would S necessarily be Dunford-Pettis? (cf. [3]). It turns out that in general the answer is in the negative. However, it turns out that if a Banach lattice E has weakly sequentially continuous lattice operations (i.e. $x_n \xrightarrow{weak} 0 \Longrightarrow |x_n| \xrightarrow{weak} 0$) we obtain:

Theorem 2.1.7 *(cf. [3], Theorem 5.89, page 345) Let $S, T : E \to F$ be two positive operators between Banach lattices such that $0 \leq S \leq T$. If E has weakly sequentially continuous lattice operations and T is Dunford-Pettis, then S is likewise Dunford-Pettis.*

Furthermore, if a positive Dunford-Pettis operator has its range in a Banach lattice with order continuous norm, then:

Theorem 2.1.8 *(cf. [3], Kalton-Saab) Let $S : E \to F$ be a positive operator between two Banach lattices such that F has order continuous norm. If S is dominated by a Dunford-Pettis operator, then S itself is Dunford-Pettis.*

2.2 Weak Dunford-Pettis operators

Aliprantis and Burkinshaw introduced the class of weak Dunford-Pettis operators (cf. [3]).

Definition 2.2.1 *(cf. [3], page 349) An operator $T : X \to Y$ between two Banach spaces is said to be a weak Dunford-Pettis operator whenever $(x_n) \in c_0^{weak}(X)$ and $(y_n^*) \in c_0^{weak}(Y^*)$ imply that $\lim_{n\to\infty}\langle Tx_n, y_n^*\rangle = 0$.*

Remark 2.2.2 *(cf. [3])*

(i) Every Dunford-Pettis operator is a weak-Dunford-Pettis operator.

(ii) If Y is reflexive, then the notions of Dunford-Pettis and weak Dunford-Pettis operator coincide.

(iii) If Y has the Dunford-Pettis property, then every continuous operator from X to Y is a weak Dunford-Pettis operator.

(iv) A weak Dunford-Pettis operator need not be a Dunford-Pettis operator.

In the paper [4], the concept of a Dunford-Pettis set is defined as follows:

Definition 2.2.3 *A norm bounded subset A of a Banach space X is a Dunford-Pettis set whenever every weakly compact operator from X to an arbitrary Banach space carries A to a norm totally bounded set.*

These sets were characterised by Andrews as follows:

Theorem 2.2.4 *(Theorem 1, [4], page 36) For a norm bounded subset A of a Banach space X the following statements are equivalent:*

(1) A is a Dunford-Pettis set.

(2) Every weakly compact operator from X to c_0 carries A to a norm totally bounded set.

(3) Every sequence $(x_n^*) \in c_0^{weak}(X^*)$ converges uniformly to zero on the set A.

We may link these sets to weak Dunford-Pettis operators as follows:

Theorem 2.2.5 *(cf. [3], page 351) For a continuous operator $T : X \to Y$ between two Banach spaces the following statements are equivalent:*

(1) T is a weak Dunford-Pettis operator.

(2) T carries weakly compact subsets of X to Dunford-Pettis subsets of Y.

(3) If S is a weakly compact operator from Y to an arbitrary Banach space, then ST is a Dunford-Pettis operator.

Since a weakly compact operator is not necessarily Dunford-Pettis, and each Dunford-Pettis operator is weak Dunford-Pettis, the question is whether each weakly compact operator is weak Dunford-Pettis. The answer is in the negative, since it is well-known that the Banach space $L^2([0,1])$ is reflexive, hence its identity operator is weakly compact, but not weak-Dunford-Pettis. Conversely a weak Dunford-Pettis operator is not always weakly compact, since the Banach space ℓ_1 has the Dunford-Pettis property, hence its identity operator is weak Dunford-Pettis, but it is not weakly compact.

Recall from [59] (Theorem 5.16, page 95) that a Banach lattice E is reflexive if and only if the norms of its topological dual E^* and of its topological bi-dual E^{**} are order continuous. We obtain the following characterisation:

Theorem 2.2.6 *([6], Theorem 2.1) Let E be a Dedekind σ-complete Banach lattice. Then the following assertions are equivalent:*

(1) E is reflexive.

(2) Each positive weak Dunford-Pettis operator from E into E is compact.

(3) For all operators $S, T : E \to E$ such that $0 \leq S \leq T$ and T is weak Dunford-Pettis, S is compact.

(4) Each positive weak Dunford-Pettis operator from E into E is weakly compact.

(5) For each positive weak Dunford-Pettis operator $T : E \to E$, the operator product T^2 is weakly compact.

Whenever $E \neq F$, we obtain the following necessary conditions:

Theorem 2.2.7 *(Theorem 2.2, [6]) Let E and F be two Dedekind σ-complete Banach lattices. If each positive weak Dunford-Pettis operator from E to F is weakly compact, then one of the following assertions is valid:*

(1) E is reflexive.

(2) F has an order continuous norm.

As a consequence of Theorem 5.24 of [3] and Theorem 2.2, we obtain the following characterisation:

Corollary 2.2.8 *(cf. [6], page 828) Let E and F be two Dedekind σ-complete Banach lattices such that F is an infinite-dimensional AM-space with unit. Then the following assertions are equivalent:*

(1) E is reflexive.

(2) Each operator from E into F is weakly compact.

(3) Each positive weak Dunford-Pettis operator from E into F is weakly compact.

Remark 2.2.9 *(cf. [6], page 828) The second necessary condition of Theorem 2.2 of [6] is not sufficient, since if we take $E = F = c_0$, then since c_0 is not reflexive but has the Dunford-Pettis property, its identity operator is weak Dunford-Pettis but not weakly compact. This despite of the fact that the norm of c_0 is order continuous.*

We however have the following property:

Corollary 2.2.10 *(cf. [6], Corollary 2.5) Let E be an infinite-dimensional AM-space with unit and F a Banach lattice. Then the following assertions are equivalent:*

(1) Each positive operator from E into F is weakly compact.

(2) Each positive weak Dunford-Pettis operator from E into F is weakly compact.

(3) The norm of F is order continuous.

Remark 2.2.11 *(cf. [6], page 829) If the Banach lattice E is an AM-space without unit, then the preceding Corollary is false. In fact, if we take $E = F = c_0$, then id_{c_0} is weak Dunford-Pettis but not weakly compact.*

The positive weak Dunford-Pettis operators also enjoy the following properties:

Theorem 2.2.12 *(cf. [3], Theorem 5.100) Let $T : E \to F$ be a positive weak Dunford-Pettis operator between Banach lattices. If $W \subseteq E$ and $V \subseteq F^*$ are two weakly relatively compact sets, then the following hold:*

(1) For every disjoint sequence $\{x_n\}$ in the solid hull of W, the sequence $\{Tx_n\}$ converges uniformly to zero on the solid hull of V.

(2) For each $\varepsilon > 0$ there exists some $u \in E^+$ satisfying

$$|f|(T(|x| - u)^+) < \varepsilon$$

for all $x \in W$ and all $f \in V$.

The following theorem shows that the weak Dunford-Pettis property of a positive operator is inherited by the positive operator it dominates.

Theorem 2.2.13 *(Kalton-Saab, cf. [3], page 353) If a positive operator S is dominated by a weak Dunford-Pettis operator, then S is a weak Dunford-Pettis operator.*

2.3 Almost Dunford-Pettis operators

Sanchez (cf. [57]) introduced the so-called almost Dunford-Pettis operator:

Definition 2.3.1 *A bounded linear operator T mapping a Banach lattice E into a Banach space is said to be almost Dunford-Pettis if $\|Tx_n\| \to 0$ for every weakly null sequence consisting of pairwise disjoint elements.*

Remark 2.3.2 *(cf. [65], page 228) T is almost Dunford-Pettis if $\|Tx_n\| \to 0$ whenever a weakly null sequence (x_n) satisfies $x_n \wedge x_m = 0$ for $n \neq m$.*

This remark is an immediate consequence of the following:

Proposition 2.3.3 *(cf. [65], page 228) If E is a Banach lattice and $(x_n) \subset E$ is a weakly null sequence with pairwise disjoint terms, then the sequences $(|x_n|)$, (x_n^+) and (x_n^-) converge weakly to zero too.*

A direct consequence of Theorem 5.57 (cf. [3], page 318) is that every almost Dunford-Pettis operator is o-weakly compact (order weakly compact). Recall that an operator T from a Banach lattice E into a Banach space X is said to be o-weakly compact if for each $x \in E^+$, the subset $T([0, x])$ is relatively weakly compact in X) (cf. [8], page 372). It is trivial that each Dunford-Pettis operator is almost Dunford-Pettis, but the converse is not true in general.

In [8], page 375 the authors give the following domination property:

Corollary 2.3.4 *Let E and F be two Banach lattices. If S and T are two positive operators from E into F such that $0 \leq S \leq T$ and T is almost Dunford-Pettis, then S is also almost Dunford-Pettis.*

Recall that the lattice operations in E^* are called weak* sequentially continuous if $(x_n^*) \in c_0^{weak*}(E^*)$ implies that $(|x_n^*|) \in c_0^{weak*}(E^*)$.

We give some sufficient conditions under which each positive almost Dunford-Pettis operator is weak Dunford-Pettis:

Corollary 2.3.5 *(cf. [7], page 173) Let E and F be two Banach lattices. Then each positive almost Dunford-Pettis operator T from E into F is weak Dunford-Pettis if one of the following assertions is valid:*

(1) The lattice operations in E are weakly sequentially continuous.

(2) F is discrete and its norm is order continuous.

(3) The topological dual F^* is discrete.

(4) The lattice operations in F^* are weakly sequentially continuous.

(5) The lattice operations in F^* are weak sequentially continuous.*

(6) E or F has the Dunford-Pettis property.

Recall the positive Schur property of a Banach lattice:

Definition 2.3.6 *(cf. [52], page 892) A Banach lattice is said to have the positive Schur property if for each* $(x_n) \in c_0^{weak}(E)$*, with* $x_n \in E^+$ *it follows that* $\|x_n\| \to 0$*.*

The following characterisation is useful:

Proposition 2.3.7 *(cf. [9], Proposition 2.1) A Banach lattice* E *does not have the positive Schur property if and only if there exists a disjoint weakly null sequence* $(x_n) \subset E^+$ *with* $\|x_n\| = 1$ *for all* n*.*

We list here a few interesting facts regarding almost Dunford-Pettis operators (cf. [65], page 229–230):

Remark 2.3.8 *(1) If* E *is a Banach lattice with the positive Schur property, then every continuous linear operator on* E *is almost Dunford-Pettis. When* E *is not discrete, then there exist almost Dunford-Pettis operators from* E *to* c_0 *which are not Dunford-Pettis.*

(2) Let E*,* X *be a Banach lattice and a Banach space respectively. If* $T : E \to X$ *factorises through a Banach lattice* F *with the positive Schur property and its factor* $S : E \to F$ *is positive, then* T *is almost Dunford-Pettis.*

(3) A bounded linear operator on $C(K)$ *is almost Dunford-Pettis if and only if it is Dunford-Pettis.*

(4) Let F *be a discrete Banach lattice with an order continuous norm. A positive operator* $T : E \to F$ *is almost Dunford-Pettis if and only if* T *is Dunford-Pettis.*

(5) Suppose E *does not contain any complemented copy of* ℓ_1 *and let* F *be a discrete Banach lattice with an order continuous norm. A positive operator* $T : E \to F$ *is almost Dunford-Pettis if and only if* T *is compact.*

(6) If E *is a discrete Banach lattice with an order continuous norm and without a complemented copy of* ℓ_1*, then an operator* T *on* E *is almost Dunford-Pettis if and only if* T *is compact.*

The next characterisation of the positive Schur property is due to Moussa and Bouras (cf.[52], page 892):

Theorem 2.3.9 *(cf. [52], Theorem 2.1) Let E be a Banach lattice. Then, the following assertions are equivalent:*

(1) Each positive operator from E into ℓ_∞ is almost Dunford-Pettis.

(2) E has the positive Schur property.

We next consider almost Dunford-Pettis operators into c_0:

Theorem 2.3.10 *(cf. [9], page 190) The following assertions about a Banach lattice E are equivalent:*

(1) Every positive operator $T : E \to c_0$ is almost Dunford-Pettis.

(2) For each weak null disjoint sequence $(x_n) \subset E^+$ and each weak null sequence $(x_n^*) \subset (E^*)^+$, we have*

$$\lim_{m\to+\infty} (\sup\{x_n^*(x_m) : n \in \mathbb{N}\}) = 0.$$

If we assume that E has order continuous norm, then we may improve Theorem 2.3.10 as follows:

Corollary 2.3.11 *(cf. [9], page 191) Let E be a Banach lattice with order continuous norm. Then the following assertions are equivalent:*

(1) Each bounded operator $T : E \to c_0$ is almost Dunford-Pettis.

(2) Each positive operator $T : E \to c_0$ is almost Dunford-Pettis.

(3) E has the positive Schur property.

Recall that a Banach lattice E is said to be a Kantorivich-Banach space (KB-space for short) whenever increasing norm bounded sequences of E^+ is norm convergent. The following theorem characterises Banach lattices for which each positive weak Dunford-Pettis operator from a Banach lattice E into a dual Banach lattice F is almost Dunford-Pettis.

Theorem 2.3.12 *(cf. [52], Theorem 2.2) Let E and F be two Banach lattices such that F is a dual Banach lattice. Then, the following assertions are equivalent:*

(1) Each positive weak Dunford-Pettis operator $T : E \to F$ is almost Dunford-Pettis.

(2) One of the following assertions is valid:

(a) E has the positive Schur property.

(b) F is a KB-space.

Remark 2.3.13 *(cf. [52], page 893) The assumption that F is a dual Banach lattice is essential in Theorem 2.3.12, since if we consider $E = \ell_\infty$ and $F = c_0$, it is clear that $F = c_0$ is not a dual Banach lattice and each operator from ℓ_∞ into c_0 is Dunford-Pettis (hence almost Dunford-Pettis). However, conditions (2)(a) and (2)(b) fail, since ℓ_∞ does not have the positive Schur property and c_0 is not a KB-space.*

The authors in [7], page 174 also asserted that:

Theorem 2.3.14 *(cf. [7], Theorem 5.3) Let E and F be two Banach lattices and let T be a positive weak Dunford-Pettis operator from E into F. Then T is almost Dunford-Pettis if one of the following assertions is valid:*

(1) F^{**} has an order continuous norm.

(2) F is a dual KB-space.

(3) F is a discrete KB-space.

(4) F has the positive Schur property.

(5) E has the positive Schur property.

If we now assume that F is Dedekind σ-complete, we obtain:

Theorem 2.3.15 *(cf. [52], Theorem 2.3) Let E and F be two Banach lattices with F Dedekind σ-complete. If each positive weak Dunford-Pettis operator from E into F is almost Dunford-Pettis, then one of the following assertions is valid:*

(1) E has the positive Schur property.

(2) F has an order continuous norm.

Remark 2.3.16 *(cf. Theorem 2.3.15)*

(1) The condition that F has an order continuous norm is not sufficient, since if we consider $E = F = c_0$, then $F = c_0$ has an order continuous norm and the identity operator id_{c_0} is weak Dunford-Pettis (since c_0 has the Dunford-Pettis property) but it is not almost Dunford-Pettis.

(2) The assumption that F is Dedekind σ-complete is essential, since if we consider $E = \ell_\infty$ and $F = c$ (the Banach lattice of all convergent sequences), then $F = c$ is not Dedekind σ-complete and each operator from ℓ_∞ into c is Dunford-Pettis (hence almost Dunford-Pettis), but conditions (1) and (2) fail since ℓ_∞ does not have the positive Schur property and the norm of c is not order continuous.

Theorem 2.3.15 has the following consequence:

Corollary 2.3.17 *(cf. [52], Corollary 2.1) Let E and F be two Banach lattices with F Dedekind σ-complete. If the norm of F is not order continuous, then the following assertions are equivalent:*

(1) Each positive operator T from E into F is almost Dunford-Pettis.

(2) E has the positive Schur property.

The next result gives some sufficient conditions for which a positive weak Dunford-Pettis operator from E into F is Dunford-Pettis:

Theorem 2.3.18 *(cf. [52], Theorem 2.4) Let E and F be two Banach lattices. Then each positive weak Dunford-Pettis operator from E into F is Dunford-Pettis if one of the following assertions is valid:*

(1) F is a dual KB-space and the lattice operations in E are weakly sequentially continuous.

(2) F is a discrete KB-space.

(3) The norm of the topological bi-dual F^{**} is order continuous and the lattice operations in E are weakly sequentially continuous.

(4) E has the Schur property.

(5) F is reflexive.

Remark 2.3.19 *All the conditions of Theorem 2.3.18 are not necessary, for instance if we take $E = \ell_\infty$ and $F = c_0$, then each bounded linear operator $T : \ell_\infty \to c_0$ is Dunford-Pettis, but c_0 is not a KB-space (respectively, the norm of the topological dual $F^{**} = \ell_\infty$ is not order continuous and ℓ_∞ does not have the Schur property).*

We may characterise the Schur property as follows:

Theorem 2.3.20 *(cf. [52], Theorem 2.5) Let E be a Banach lattice. Then, the following assertions are equivalent:*

(1) Each positive operator from E into ℓ_∞ is Dunford-Pettis.

(2) E has the Schur property.

As a converse to this theorem, we have:

Theorem 2.3.21 *(cf. [52], Theorem 2.6) Let E and F be two Banach lattices such that F is Dedekind σ-complete. If each positive weak Dunford-Pettis operator from E into F is Dunford-Pettis, then one of the following assertions is valid:*

(1) E has the Schur property.

(2) F has an order continuous norm.

Remark 2.3.22 *(cf. Theorem 2.3.21)*

(1) The condition that F has an order continuous norm is not sufficient, since if we consider $E = F = c_0$, then $F = c_0$ has an order continuous norm and the identity operator id_{c_0} is weak Dunford-Pettis (since c_0 has the Dunford-Pettis property) but it is not Dunford-Pettis.

(2) The assumption that F is Dedekind σ-complete is essential, since if we consider $E = \ell_\infty$ and $F = c$, then $F = c$ is not Dedekind σ-complete and each operator from ℓ_∞ into c is Dunford-Pettis, but conditions (1) and (2) fail since ℓ_∞ does not have the Schur property and the norm of c is not order continuous.

If we assume that E has an order continuous norm, then:

Theorem 2.3.23 *(cf. [52], Theorem 2.7) Let E and F be two Banach lattices such that E has an order continuous norm. If each positive weak Dunford-Pettis operator T from E into F is Dunford-Pettis, then one of the following assertions is valid:*

(1) E has the Schur property.

(2) F has an order continuous norm.

Theorem 2.3.23 aids in the following characterisation:

Corollary 2.3.24 *(cf. [52], page 895) Let E and F be two Banach lattices such that the norm of F is not order continuous. Then the following assertions are equivalent:*

(1) E has an order continuous norm and each positive operator T from E into F is Dunford-Pettis.

(2) E has the Schur property.

By modifying the statement of Theorem 2.3.23 slightly, we obtain the following:

Theorem 2.3.25 *(cf. [52], Theorem 2.8) Let E and F be two Banach lattices such that E has an order continuous norm. If each weak Dunford-Pettis operator T from E into F is Dunford-Pettis, then one of the following assertions is valid:*

(1) E has the Schur property.

(2) F is a KB-space.

Remark 2.3.26 *The second necessary condition of Theorem 2.3.25 is not sufficient, since if we take $E = F = L_1[0,1]$, then since $L_1[0,1]$ is a KB-space, its identity operator $id_{L_1[0,1]}$ is weak Dunford-Pettis but not Dunford-Pettis.*

As a consequence of Theorem 2.3.25 we obtain:

Corollary 2.3.27 *(cf. [52], page 896) Let E and F be two Banach lattices such that F is not a KB-space. Then the following assertions are equivalent:*

(1) E has an order continuous norm and each operator T from E into F is Dunford-Pettis.

(2) E has the Schur property.

Note that in Corollary 2.3.27 we cannot replace "each operator" with "each positive operator", since if we take $E = L_1[0,1]$ and $F = c_0$, then the Banach lattice $L_1[0,1]$ does not have the Schur property, but $L_1[0,1]$ has an order continuous norm and each operator T from E into F is Dunford-Pettis.

Räbiger (cf. [56]) introduced the so-called weak Dunford-Pettis property:

Definition 2.3.28 *(cf. [65], page 230) A Banach lattice E is said to have the weak Dunford-Pettis property if every weakly compact operator on E is almost Dunford-Pettis.*

It is trivial to see that if E has the Dunford-Pettis property or E has the Schur property (i.e. each weakly null sequence in E converges to zero in norm), then E has the weak Dunford-Pettis property.

The following corollary gives a sufficient condition under which the weak Dunford-Pettis property and the Dunford-Pettis property coincide:

Corollary 2.3.29 *(cf. [8], Corollary 2.8) Let E be a Banach lattice with weak sequentially continuous lattice operations. Then E has the weak Dunford-Pettis property if and only if it has the Dunford-Pettis property.*

The following characterisation is useful:

Proposition 2.3.30 *(cf. [65], Proposition 1) For a Banach lattice E the following statements are equivalent:*

(i) E has the weak Dunford-Pettis property.

(ii) Every weakly compact operator from E to c_0 is almost Dunford-Pettis.

(iii) If a sequence $(x_n) \subset E^+$ is weakly null and it has pairwise disjoint terms and $(x_n^*) \in c_0^{weak}(E^*)$, then $x_n^*(x_n) \to 0$.

2.4 Weak* Dunford-Pettis operators

Recall the following definitions from the literature:

Definition 2.4.1 *(1) A norm bounded subset A of a Banach space X is said to be a limited set if every weak* null sequence (x_n^*) of X^* converges uniformly on A, i.e. $\lim_{n\to\infty}\sup_{x\in A}|x_n^*(x)| = 0$.*

(2) If all limited sets in a Banach space X are relatively compact, then X is said to have the Gelfand-Phillips property.

(3) A Banach space X is said to have the DP property if every relatively weakly compact subset of X is limited (cf. [19]).*

(4) X has the DP property if and only if $\lim_{n\to\infty} x_n^*(x_n) = 0$ for every $(x_n) \in c_0^{weak}(X)$ and every $(x_n^*) \in c_0^{weak^*}(X^*)$.*

In [51], the authors introduced the following class of operators:

Definition 2.4.2 *A bounded linear operator $T : X \to Y$ between two Banach spaces is said to be a weak* Dunford-Pettis operator (abbreviated hereafter as w*DP) whenever $x_n \to 0$ in the $\sigma(X, X^*)$-topology of X and $y_n^* \to 0$ in the $\sigma(Y^*, Y)$-topology of Y^* imply $\lim_{n\to\infty} y_n^*(Tx_n)) = 0$.*

Remark 2.4.3 *(cf. [51], page 3)*

*(a) The class of w*DP operators is bigger than the class of Dunford-Pettis operators, but smaller than the class of weak Dunford-Pettis operators.*

*(b) id_{ℓ_∞} is w*DP (since ℓ_∞ has DP* property), but is not a Dunford-Pettis operator (since ℓ_∞ does not have the Schur property).*

*(c) id_{c_0} is weak Dunford-Pettis (since c_0 has the DP property), but is not w*DP (since c_0 does not have the DP* property).*

(d) Recall that in a Grothendieck space, weak and weak continuity coincides, hence if Y is a Grothendieck space, then the notions of weak Dunford-Pettis and w*DP operators coincide.*

Proposition 2.4.4 *(cf. [51]) Let X be a Banach space. Then the following statements are equivalent:*

(1) X has the DP property.*

*(2) Every bounded linear operator $T : X \to X$ is w*DP.*

*(3) The identity operator of X is w*DP.*

Consequently,

Corollary 2.4.5 *(cf. [51], Corollary 2.1) Let X be a Banach space. Then the following statements are equivalent:*

(1) X has the DP property.*

*(2) For an arbitrary Banach space Y, every bounded linear operator $T : X \to Y$ is w*DP.*

*(3) For an arbitrary Banach space Y, every bounded linear operator $T : Y \to X$ is w*DP.*

Recall from [46] that a bounded linear operator T from a Banach space X into another Y is said to be a limited operator if it carries $\mathcal{B}_X$ into a limited set of Y. An operator T is limited if and only if T^* takes weak* null sequences to norm null ones.

The authors in [46] (page 263) give the following necessary and sufficient condition under which each operator is w*DP:

Theorem 2.4.6 *(cf. [46], Theorem 3.2) Let X and Y be two Banach spaces and let T be an operator from X into Y. Then the following assertions are equivalent:*

*(1) T is w*DP.*

(2) T carries weakly compact subsets of X to limited subsets of Y.

(3) If S is a weakly compact operator from an arbitrary Banach space Z to X, then $T \circ S$ is a limited operator.

(4) If S is a weakly compact operator from ℓ_1 to X, then $T \circ S$ is a limited operator.

Consequently,

Corollary 2.4.7 *(cf. [46], page 264) Let X be a Banach space. Then the following assertions are equivalent:*

(1) X has the DP property.*

*(2) id_X is w*DP, i.e. every relatively weakly compact set of X is a limited set.*

(3) every weakly compact operator T from an arbitrary Banach space Z to X is a limited operator.

(4) every weakly compact operator $T : \ell_1 \to X$ is a limited operator.

Recall from [46] that a bounded linear operator $T : X \to Y$ between two Banach spaces is said to be limited completely continuous (lcc for short) if it carries sequences in X which are both limited and weakly null to norm null sequences in Y.

Remark 2.4.8 *(1) There exists a w*DP operator which is not lcc.*

*(2) id_{ℓ_∞} is w*DP (since ℓ_∞ has the DP* property), but is not an lcc operator (since ℓ_∞ does not have the Gelfand-Phillips property (GP-property for short))*

The authors in [46] give the following characterisation in terms of Banach lattices:

Theorem 2.4.9 *(cf. [46], Theorem 3.4) Let E and F be two Banach lattices such that F is Dedekind σ-complete. Then the following assertions are equivalent:*

*(1) each w*DP operator T from E into F is lcc.*

(2) one of the following is valid:

(a) E has the GP-property.

(b) F has an order continuous norm.

Note that each operator $T : \ell_\infty \to c$ is weakly compact, hence lcc, but ℓ_∞ does not have the GP-property and c does not have an order continuous norm.

As a consequence of Theorem 2.4.9, we have the following characterisations:

Corollary 2.4.10 *(cf. [46], Corollary 3.5) Let E be a Banach lattice. Then the following assertions are equivalent:*

*(1) each w*DP operator T from E into ℓ_∞ is lcc.*

(2) E has the GP-property.

Corollary 2.4.11 *(cf. [46], Corollary 3.6) Let F be a Banach lattice such that F is Dedekind σ-complete. Then the following assertions are equivalent:*

*(1) each w*DP operator T from E into ℓ_∞ is lcc.*

(2) F has an order continuous norm.

Chen et. al. noted that in a Banach lattice (or its dual) the lattice operations fail to be weakly (respectively, weak*) sequentially continuous in general and that every disjoint sequence in the solid hull of a relatively weakly compact subset of a Banach lattice E converges weakly to zero (cf. [22], page 312). They also remarked that if (x_n) is a disjoint, weakly convergent sequence in E, then the sequences (x_n), $(|x_n|)$, (x_n^+) and (x_n^-) are all weakly convergent to zero. In Example 2.1 of [21] the authors demonstrated that the weak* disjoint sequences in the dual space do not necessarily enjoy such a property. However, for a σ-Dedekind complete Banach lattice, we have the following:

Lemma 2.4.12 *(cf. [21], Lemma 2.2) Let E be a σ-Dedekind complete Banach lattice and let (x_n^*) be a weak* convergent sequence in E^*. If (y_n^*) is a disjoint sequence of E^* satisfying $|y_n^*| \leq |x_n^*|$ for each $n \in \mathbb{N}$, then the sequences (y_n^*), $(|y_n^*|)$, $((y_n^*)^+)$ and $((y_n^*)^-)$ are all weak* convergent to zero. In particular, if (x_n^*) is a disjoint weak* convergent sequence, then the sequences (x_n^*), $(|x_n^*|)$, $((x_n^*)^+)$ and $((x_n^*)^-)$ are all weak* null.*

If the range space is σ-Dedekind complete, we obtain the following set of theorems:

Theorem 2.4.13 *(cf. [22], Theorem 2.2) Let $T : E \to F$ be a positive weak* Dunford-Pettis operator between Banach lattices E and F with F σ-Dedekind complete. Then, for every weakly null sequence (x_n) in E^+ and every weak* null sequence (y_n^*) in F^*, we have $|y_n^*|(Tx_n) \to 0$ (as $n \to \infty$).*

Theorem 2.4.14 *(cf. [22], Theorem 2.3) Let $T : E \to F$ be a positive weak* Dunford-Pettis operator between Banach lattices E and F with F σ-Dedekind complete. Let W be a relatively weakly compact subset of E and (y_n^*) be a weak* null sequence in F^*. Then, for any $\varepsilon > 0$, there exist some $N \in \mathbb{N}$ and some $u \in E^+$ lying in the ideal generated by W such that*

$$|y_n^*|(T(|x|) - u)^+ < \varepsilon$$

for all $n > N$ and all $x \in W$.

Theorem 2.4.15 *(cf. [22], Theorem 3.1) Let E and F be two Banach lattices with F σ-Dedekind complete. If a positive operator $S : E \to F$ is dominated by a positive weak* Dunford-Pettis operator, then S is also weak* Dunford-Pettis.*

Chapter 3

On p-convergent operators and some variants

The notion of a p-convergent operator on a Banach space is introduced in the paper [20] and plays an important role in the study of the DP^*-property of order p (cf. [37]). In this chapter we consider the p-convergent operators on Banach lattices, prove some ideal properties of the same and consider their applications (in combination with the notion of a weak p-convergent operator, also introduced in this chapter) to a study of the Schur property of order p. Also, the notion of a disjoint p-convergent operator on Banach lattices is introduced, studied and its applications to a study of the positive Schur property of order p are considered. Please note that the main results in this chapter was submitted for publication (cf. [68]).

3.1 Disjoint p-convergent operators and weak p-convergent operators

Throughout this section we assume that $1 \leq p < \infty$, unless otherwise stated. We recall some notions from the paper [20].

Definition 3.1.1 *(cf. [20]) Let X be a Banach space. A sequence $(x_i) \subset X$ is said to be weakly p-convergent to $x \in X$ if $(x_i - x) \in \ell_p^{weak}(X)$. A bounded subset $A \subset X$ is said to be relatively weakly p-compact if every sequence in A has a weakly p-convergent subsequence.*

The following family of operators was introduced in the paper [20] and applied in the paper [37] to study the DP^*-property of order p in Banach spaces:

Definition 3.1.2 *(cf. [20]) An operator T from a Banach space X into a Banach space Y is said to be p-convergent if the sequence $(\|Tx_n\|)$ converges to 0 in Y, for every sequence $(x_n) \in \ell_p^{weak}(X)$. This is equivalent to the statement that T is p-convergent if it maps relatively weakly p-compact sets of X into relatively norm compact sets of Y.*

It follows easily from the definition that:

Proposition 3.1.3 *(a) If $S : X \to Y$ is p-convergent, $T \in \mathcal{L}(X_0, X)$, $R \in \mathcal{L}(Y, Y_0)$, then $RST : X_0 \to Y_0$ is p-convergent.*

(b) Suppose that $T \in \mathcal{L}(X, Y)$ and $J : Y \to Z$ is an isomorphism from Y into Z. Then, if JT is p-convergent, then so is T.

Remark 3.1.4 *Let (x_i) be a sequence in a Banach lattice E, such that for all $(\lambda_i) \in \ell_p$ the corresponding series $\sum_{i=1}^{\infty} \lambda_i x_i$ converges in E. Then, since for each $(\lambda_i) \in \ell_p$, also $(|\lambda_i|) \in \ell_p$, it follows that $(\lambda_i^+), (\lambda_i^-) \in \ell_p$. Thus, the series $\sum_{i=1}^{\infty} |\lambda_i| x_i$, $\sum_{i=1}^{\infty} \lambda_i^+ x_i$ and $\sum_{i=1}^{\infty} \lambda_i^- x_i$ converge in E as well.*

Proposition 3.1.5 *Let E be a Banach lattice and suppose that (x_i) is a sequence in E such that the elements x_i are pairwise disjoint. Then we have*

$$(x_i) \in \ell_p^{weak}(E) \iff (|x_i|) \in \ell_p^{weak}(E).$$

Proof For $1 \leq p < \infty$, let $(x_i) \in \ell_p^{weak}(E)$. Then $\sum_{i=1}^{\infty} \lambda_i x_i$ converges in E for each $(\lambda_i) \in \ell_{p'}$, where we let $\ell_{1'} = c_0$. By Remark 3.1.4 above, both $\sum_{i=1}^{\infty} \lambda_i^+ x_i$ and $\sum_{i=1}^{\infty} \lambda_i^- x_i$ converge in E. The elements of (x_i) being pairwise disjoint, it follows for all $1 \leq m < n$ that:

$$\begin{aligned} \sum_{i=m+1}^{n} \lambda_i |x_i| &= \sum_{i=m+1}^{n} \lambda_i^+ |x_i| - \sum_{i=m+1}^{n} \lambda_i^- |x_i| \\ &= \sup_{m+1 \leq i \leq n} (\lambda_i^+ |x_i|) - \sup_{m+1 \leq i \leq n} (\lambda_i^- |x_i|) \\ &= \left| \sum_{i=m+1}^{n} \lambda_i^+ x_i \right| - \left| \sum_{i=m+1}^{n} \lambda_i^- x_i \right|. \end{aligned}$$

Thus, $\sum_{i=1}^{\infty} \lambda_i |x_i|$ converges in E for all $(\lambda_i) \in \ell_{p'}$, showing that $(|x_i|) \in \ell_p^{weak}(E)$.

Conversely, if $(|x_i|) \in \ell_p^{weak}(E)$, then for each $(\lambda_i) \in \ell_{p'}$ the series $\sum_{i=1}^{\infty} \lambda_i |x_i|$ converges in E. Therefore, since also $(|\lambda_i|) \in \ell_{p'}$, the series $\sum_{i=1}^{\infty} |\lambda_i||x_i|$ converges in E for all $(\lambda_i) \in \ell_{p'}$. It follows from

$$\begin{aligned} \left| \sum_{i=1}^{\infty} \lambda_i x_i \right| &= \sup_i |\lambda_i||x_i| \\ &= \sum_{i=1}^{\infty} |\lambda_i||x_i|, \end{aligned}$$

that $\sum_{i=1}^{\infty} \lambda_i x_i$ converges for all $(\lambda_i) \in \ell_{p'}$, i.e. that $(x_i) \in \ell_p^{weak}(E)$. □

It follows from Proposition 3.1.5 and $x_n^+ \leq |x_n|$ and $x_n^- \leq |x_n|$ for all n, that:

Corollary 3.1.6 *Let E be a Banach lattice and suppose that the elements x_i of a sequence $(x_i) \subset E$ are pairwise disjoint. Then we have*

$$(x_i) \in \ell_p^{weak}(E) \iff (x_i^+), (x_i^-) \in \ell_p^{weak}(E).$$

Definition 3.1.7 *Let E, F be Banach lattices. An operator $T : E \to F$ is said to be disjoint p-convergent if $\|Tx_n\| \to 0$ for all weakly p-summable sequences (x_n) so that the elements x_i are pairwise disjoint.*

Suppose an operator $T : E \to F$ satisfies $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ such that $x_n \wedge x_m = 0$ if $m \neq n$. Now assume that $(y_n) \in \ell_p^{weak}(E)$ and the elements y_i are pairwise disjoint, i.e. $|y_m| \wedge |y_n| = 0$ for $m \neq n$. Then $y_n^+ \wedge y_m^+ = 0$ and $y_n^- \wedge y_m^- = 0$ for $m \neq n$ and by Corollary 3.1.6 we have $(y_n^+) \in \ell_p^{weak}(E)$ and $(y_n^-) \in \ell_p^{weak}(E)$. Therefore, both $\|Ty_n^+\| \to 0$ and $\|Ty_n^-\| \to 0$ as $n \to \infty$, and it follows that

$$\|Ty_n\| = \|Ty_n^+ - Ty_n^-\| \leq \|Ty_n^+\| + \|Ty_n^-\| \to 0 \text{ if } n \to \infty.$$

We may therefore conclude that:

Proposition 3.1.8 *Let E, F be Banach lattices. If an operator $T : E \to F$ satisfies $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ such that $x_n \wedge x_m = 0$ for $m \neq n$, then T is disjoint p-convergent.*

Moreover, the above argument also shows that:

Proposition 3.1.9 *Let E, F be Banach lattices. An operator $T : E \to F$ is disjoint p-convergent if and only if $\|Tx_n\| \to 0$ for all sequences $(x_n) \in \ell_p^{weak}(E)$ consisting of pairwise disjoint positive elements.*

The disjoint p-convergent operators satisfy the following "domination property":

Theorem 3.1.10 *If E, F are Banach lattices and $S, T : E \to F$ are positive linear operators with $0 \leq S \leq T$ and T disjoint p-convergent, then S is also disjoint p-convergent.*

Proof Let (x_n) be weakly p-summable with $|x_m| \wedge |x_n|$ if $m \neq n$. Then we have

$$|Sx_n| \leq S|x_n| \leq T|x_n|$$

for all n. Now, by Proposition 3.1.5 we have $(|x_n|) \in \ell_p^{weak}(E)$. Thus, $\|T|x_n|\,\| \to 0$ as $n \to \infty$. Therefore,

$$\|Sx_n\| = \|\,|Sx_n|\,\| \leq \|S|x_n|\,\| \leq \|T|x_n|\,\| \to 0 \text{ as } n \to \infty,$$

showing that S is disjoint p-convergent as well. □

The introduction of "weak Dunford-Pettis operator" in the paper [52], motivates the following notion of "weak p-convergent operator" on Banach spaces:

Definition 3.1.11 *An operator $T : X \to Y$ is said to be weak p-convergent if $y_n^*(Tx_n) \xrightarrow[\infty]{n} 0$ for every $(x_n) \in \ell_p^{weak}(X)$ and every $(y_n^*) \in c_0^{weak}(Y^*)$.*

Recall from [20] (page 50) that a Banach space X is said to have the Dunford-Pettis property of order p (DPP_p for short) if for $(x_n) \in \ell_p^{weak}(X)$ and $(x_n^*) \in c_0^{weak}(X^*)$ we have $\langle x_n^*, x_n \rangle \to 0$ as $n \to \infty$. Clearly this is equivalent to saying that id_X is weak p-convergent.

The relationship between weak p-convergent and p-convergent operator is described by the following proposition:

Proposition 3.1.12 *Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$. The following statements are equivalent:*

(i) T is weak p-convergent.

(ii) ST is p-convergent for each weakly compact operator $S : Y \to Z$ and any Banach space Z.

(iii) ST is p-convergent for each weakly compact operator $S : Y \to c_0$.

Proof $(i) \implies (ii)$: Suppose the implication is not true and let $T : X \to Y$ be a weak p-convergent operator for which there exist a Banach space Z and and a $S \in \mathcal{W}(Y,Z)$ such that ST is not p-convergent. Then there exists $(x_n) \in \ell_p^{weak}(X)$ such that $\|(ST)x_n\| \not\to 0$ as $n \to \infty$. We may assume (by taking a subsequence if necessary) that there is an $\varepsilon > 0$ such that $\|(ST)x_n\| \geq \varepsilon$ for all $n \in \mathbb{N}$. Again, by taking a subsequence, we may assume that there exists $z_n^* \in \mathcal{B}_{Z^*}$ such that $|\langle z_n^*, (ST)x_n\rangle| \geq \frac{\varepsilon}{2}$ for all $n \in \mathbb{N}$. Now $(S^* z_n^*) \subset S^*(\mathcal{B}_{Z^*})$, where $S^*(\mathcal{B}_{Z^*})$ is relatively weakly compact. Therefore, there exists a subsequence $(z_{n_k}^*)$ such that $S^* z_{n_k}^* \xrightarrow[\infty]{k} y^* \in Y^*$ weakly. Thus, by assumption,

$$\langle y^*, Tx_{n_k}\rangle - \langle S^* z_{n_k}^*, Tx_{n_k}\rangle \xrightarrow[\infty]{k} 0.$$

However, since $Tx_{n_k} \xrightarrow[\infty]{k} 0$ weakly, this implies that

$$\langle S^* z_{n_k}^*, Tx_{n_k}\rangle \xrightarrow[\infty]{k} 0,$$

thereby contradicting that $|\langle z_{n_k}^*, (ST)x_{n_k}\rangle| \geq \frac{\varepsilon}{2}$ for all $k \in \mathbb{N}$.
$(ii) \implies (iii)$: This is trivial.
$(iii) \implies (i)$: Let $(x_n) \in \ell_p^{weak}(X)$ and $(y_n^*) \in c_0^{weak}(Y^*)$, and let $S = L_{(y_i^*)}$ (as is discussed in the introduction). Then $S : Y \to c_0$ is weakly compact and

$$\begin{aligned} \|(ST)x_n\| &= \sup_j |\langle e_j, S(Tx_n)\rangle| \\ &= \sup_j |\langle e_j, (\langle y_i^*, Tx_n\rangle)_i\rangle| \\ &= \sup_j |\langle y_j^*, Tx_n\rangle|. \end{aligned}$$

Because of $|\langle y_n^*, Tx_n\rangle| \leq \sup_j |\langle y_j^*, Tx_n\rangle|$ for all n and ST being p-convergent, we have $\langle y_n^*, Tx_n\rangle \xrightarrow[\infty]{n} 0$. The choice of $(x_n) \in \ell_p^{weak}(X)$ and $(y_n^*) \in c_0^{weak}(Y^*)$ being arbitrary, it follows that T is weak p-convergent. □

Clearly, each p-convergent operator is weak p-convergent. From Proposition 3.1.12, taking $S = id_Y$, it follows that

Corollary 3.1.13 *If Y is a reflexive Banach space, then each weak p-convergent operator from any Banach space to Y is p-convergent.*

The following properties are easily verified:

Proposition 3.1.14 *Let X_0, X, Y, Z be Banach spaces. Then:*

(a) Let $T : X \to Y$ and $S \in \mathcal{L}(Y, Z)$. If T is weak p-convergent, then $ST : X \to Z$ is weak p-convergent.

(b) Let $T : X \to Y$ and $S \in \mathcal{L}(X_0, X)$. If T is weak p-convergent, then $TS : X_0 \to X$ is weak p-convergent.

Remark 3.1.15 *(a) By definition, each p-convergent operator between Banach lattices is disjoint p-convergent.*
(b) Weak p-convergent operators between Banach lattices differ from disjoint p-convergent operators: For instance, c_0 has DPP_p (since c_0 has DPP); thus id_{c_0} is weak p-convergent. However, since $(e_n) \in \ell_p^{weak}(c_0)$, (e_n) is a disjoint sequence in c_0 and $\|e_n\| \nrightarrow 0$, it follows that id_{c_0} is not disjoint p-convergent.
(c) Suppose a Banach space Y has DPP_p. Let X be any Banach space and $T : X \to Y$ a bounded linear operator. Consider $(x_n) \in \ell_p^{weak}(X)$. Then $(Tx_n) \in \ell_p^{weak}(Y)$. Therefore, if $(y_n^*) \subset Y^*$, $y_n^* \to 0$ weakly, then $y_n^*(Tx_n) \xrightarrow[\infty]{n} 0$, so T is weak p-convergent.

For proofs of the following Lemmas, the reader is referred to [32].

Lemma 3.1.16 *(cf. [32], Corollary 2.6, page 296) Let E be a Banach lattice and let (x_n) be a sequence of E. Then the following statements are equivalent:*

(1) $\|x_n\| \to 0$.

(2) $|x_n| \to 0$ weakly and $f_n(x_n) \to 0$ for every bounded disjoint sequence (f_n) in $(E^*)^+$.

Lemma 3.1.17 *(cf. [32], Corollary 2.7, page 297) Let E be a Banach lattice and let (f_n) be a sequence of E^*. Then the following statements are equivalent:*

(1) $\|f_n\| \to 0$.

(2) $|f_n| \to 0$ *weak** *and* $f_n(x_n) \to 0$ *for every bounded disjoint sequence* (x_n) *in* E^+.

Lemma 3.1.18 *(cf. [32], Corollary 2.9, page 297) Let E be a Banach lattice. Then the following statements are equivalent:*

(1) The norm of E^* is order continuous.

(2) Each disjoint norm-bounded sequence in E is $\sigma(E, E^*)$ convergent to 0.

We introduce the notion of "positive Schur property of order p" as follows:

Definition 3.1.19 *A Banach lattice E is said to have the positive Schur property of order p (briefly, E has the SP_p^+) if each sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, is norm convergent to 0.*

If we agree to say that E has the SP_∞^+ if each sequence $(x_n) \in c_0^{weak}(E)$ with positive terms, is norm convergent to 0, then we may assume $1 \leq p \leq \infty$ in Definition 3.1.19; the SP_∞^+, however, will then coincide with the well known positive Schur property, which was considered in Chapter 2.

Suppose that in a Banach lattice E there exists a sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, which is not norm convergent to 0. Thus, we may assume that $c := \inf_n \|x_n\| > 0$. Then, putting $y_n = c^{-1}x_n$ for all n and using Corollary 5 in [42], we find a subsequence (y_{nk}), a constant $d > 0$, and a sequence (z_k) of pairwise disjoint elements such that $0 < z_k \leq y_{n_k}$ and $\|z_k\| \geq d$ for all k. The sequence (y_n) belongs to $\ell_p^{weak}(E)$ and therefore, by Remark 1.1.1, also $(z_k) \in \ell_p^{weak}(E)$. This proves the following:

Lemma 3.1.20 *If in a Banach lattice E there exists a sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, which is not norm convergent to 0, then there exists a sequence $(z_k) \in \ell_p^{weak}(E)$ such that $z_n \geq 0$ for all n, $z_n \wedge z_m = 0$ for all $m \neq n$ and $\|z_n\| \nrightarrow 0$.*

It therefore follows that:

Proposition 3.1.21 *A Banach lattice E has the positive Schur property of order p if and only if each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ with positive terms, is norm convergent to 0. This is equivalent to id_E being disjoint p-convergent (cf. Proposition 3.1.9).*

Theorem 3.1.22 *Let E be a Banach lattice and $1 \leq p < \infty$. Then, the following statements are equivalent:*

(1) Each positive operator from E to ℓ_∞ is disjoint p-convergent.

(2) E has the SP_p^+.

Proof (1) $\implies$ (2) : We follow the argument in the proof of Theorem 2.1 of [52], adjusted to our situation. Suppose E does not have the SP_p^+. Then there exists a disjoint weak p-summable sequence $(x_n) \subset E^+$ which is not norm convergent to 0. It follows from Lemma 3.1.16, that there exists a bounded disjoint sequence (f_n) in $(E^*)^+$ such that $f_n(x_n) \geq \varepsilon$ for every $n \in \mathbb{N}$ and some fixed $\varepsilon > 0$. Consider the operator $S : E \to \ell_\infty$ defined by

$$S(x) = (f_m(x))_{m \in \mathbb{N}} \text{ for each } x \in E.$$

Note that (x_n) is a disjoint weakly p-summable sequence in E, and

$$\|S(x_n)\| = \|(f_m(x_n))_{m \in \mathbb{N}}\|_\infty \geq f_n(x_n) \geq \varepsilon, \ \forall n.$$

This presents a contradiction, since S is positive and not disjoint p-convergent. Hence E has SP_p^+.
(2) $\implies$ (1) : This is trivial. □

The reader is referred to the book [3] to recall the definitions of the following well-known notions:
(a) Banach lattice with order continuous norm (cf. [3], page 186),
(b) Kantorovich-Banach space, or briefly a KB-space (cf. [3], page 232),
(c) Dual KB-space (cf. [3], page 232).

It is important to note that reflexive Banach lattices and AL-spaces are examples of KB-spaces (cf. [3], page 232) and that KB-spaces have order continuous norms. Banach lattices with order continuous norms are not necessarily KB-spaces (cf. [59], page 290), however the topological dual space E^* of a Banach lattice E is a KB-space if and only if its norm is order continuous (cf. [3], Theorem 4.59).

Recall that a linear operator $T : X \to Y$ between two Banach spaces is said to be an embedding whenever there exist two positive constants K and M satisfying $K\|x\| \leq \|Tx\| \leq M\|x\|$ for all $x \in X$. In this case $T(X)$ is topologically identified with X; we call $T(X)$ a copy of X in Y and X is said to be embeddable into Y. When an embedding $T : E \to F$ between two Banach lattices is also a lattice homomorphism, then T is called a lattice embedding and we call E lattice embeddable into F.

The reader is referred to Theorem 4.69 of [3] where it is proven (among others) that for a Banach lattice E the dual space E^* is not a KB-space if and only if ℓ_∞ embeds complementably in E^*. Therefore, it follows in particular that if $F = E^*$ for some Banach lattice E, i.e. if F is a dual Banach lattice, then F is not a KB-space if and only if ℓ_∞ embeds complementably in F. With this in mind, we have the following result:

Proposition 3.1.23 *Let E, F be Banach lattices such that F is a dual Banach lattice which is not a KB-space. Suppose that $J : \ell_\infty \hookrightarrow F$ denotes the complemented embedding of ℓ_∞ into F and that $P : F \to J(\ell_\infty)$ denotes the projection from F onto its complemented subspace $J(\ell_\infty)$. Then:*

(a) $T : E \to \ell_\infty$ is weak p-convergent if and only if $J \circ T : E \to F$ is weak p-convergent.

(b) $T : E \to \ell_\infty$ is disjoint p-convergent if and only if $J \circ T : E \to F$ is disjoint p-convergent.

Proof (a) If $T : E \to \ell_\infty$ is weak p-convergent, then JT is weak p-convergent.

Conversely, suppose that $J \circ T$ is weak p-convergent. Choose arbitrary $(x_n) \in \ell_p^{weak}(E)$ and $(z_n^*) \in c_0^{weak}(\ell_\infty^*)$. Let $y_n^* = z_n^* \circ J^{-1} \circ P$ for all $n \in \mathbb{N}$; then $(y_n^*) \subset F^*$ and $y_n^* = (P^* \circ (J^{-1})^*)(z_n^*)$. For each $\phi \in F^{**}$ we have $P^{**}(\phi) \in J(\ell_\infty)^{**}$ and $\theta := (J^{-1})^{**}(P^{**}(\phi)) \in \ell_\infty^{**}$, where

$$\langle \theta, z_n^* \rangle = \langle \phi, z_n^* \circ J^{-1} \circ P \rangle$$

for all $z_n^* \in \ell_\infty^*$. Thus we have $\langle \phi, y_n^* \rangle = \langle \theta, z_n^* \rangle \xrightarrow[\infty]{n} 0$ for all $\phi \in F^{**}$. Our assumption implies that $\langle y_n^*, (J \circ T)x_n \rangle \xrightarrow[\infty]{n} 0$. Since $J^{-1} \circ P \circ J = id_{\ell_\infty}$, it follows that

$$z_n^*(Tx_n) = z_n^*((J^{-1} \circ P \circ J)Tx_n) = y_n^*(J(Tx_n)) \xrightarrow[\infty]{n} 0.$$

(b) Suppose $T : E \to \ell_\infty$ is disjoint p-convergent. Then $J \circ T : E \to F$ is disjoint p-convergent.

Conversely, suppose $J \circ T : E \to F$ is disjoint p-convergent. J being an embedding, there exists $K > 0$ such that for any disjoint sequence $(x_n) \subset E^+$ with $(x_n) \in \ell_p^{weak}(E)$ we have

$$\|Tx_n\|_\infty \leq K\|J(Tx_n)\|_F \xrightarrow[\infty]{n} 0.$$

This proves that T is disjoint p-convergent. □

In the paper [52] several results concerning the positive Schur property are discussed. Most of these results can be carried over to the setting of the positive Schur property of order p, using similar (or, sometimes the same) arguments, adjusted to fit into the current context. We discuss some results.

Theorem 3.1.24 *Let E and F be two Banach lattices such that F is a dual Banach lattice. Then the following assertions are equivalent:*

(1) Each positive weak p-convergent operator $T : E \to F$ is disjoint p-convergent.

(2) One of the following assertions is valid:

(a) E has the SP_p^+.

(b) F is a KB-space.

Proof (1) ⇒ (2) : It is enough to show that if F is not a KB-space, then E has the SP_p^+. Suppose that F is not a KB-space and consider an arbitrary positive operator $T : E \to \ell_\infty$. Since ℓ_∞ is order embeddable in F (Theorem 4.69, [3]) and ℓ_∞ has DPP_p, T is weak p-convergent by Remark 3.1.15(c). By Proposition 3.1.23(a), $J \circ T$ is weak p-convergent, where $J : \ell_\infty \hookrightarrow F$ is the embedding. Our assumption implies that $J \circ T$ is disjoint p-convergent. Finally, Proposition 3.1.23(b) and Theorem 3.1.22 imply that E has SP_p^+.
(2)(a) ⇒ (1) : Obvious.

$(2)(b) \Rightarrow (1)$: Let $F = F_0^*$ for some Banach lattice F_0. For any disjoint sequence $(x_n) \subset E^+$ such that $(x_n) \in \ell_p^{weak}(E)$, we use Lemma 3.1.17 to prove that $\|Tx_n\| \xrightarrow[\infty]{n} 0$ if $T : E \to F$ is any positive weak p-convergent operator.

First observe that for each $y \in F_0$ we have

$$\langle |Tx_n|, y\rangle = \langle Tx_n, y\rangle \xrightarrow[\infty]{n} 0,$$

since T is positive and $(x_n) \in c_0^{weak}(E)$. Thus $(|Tx_n|) \in c_0^{weak^*}(F_0^*)$.

Now, suppose that $(z_n) \subset F_0$ is a bounded disjoint sequence in F_0^+. By Lemma 3.1.18, the sequence (z_n) in the space F_0 is $\sigma(F_0, F_0^*)$-convergent to 0, since F_0^* has order continuous norm.
Now the embedding $\tau : F_0 \hookrightarrow F_0^{**}$ is weakly continuous and so $(\tau z_n) \in c_0^{weak}(F_0^{**})$, i.e. $\tau z_n \to 0$ in the $\sigma(F_0^{**}, (F_0^{**})^*) = \sigma(F^*, F^{**})$-topology. This implies that

$$\langle Tx_n, z_n\rangle = \langle \tau z_n, Tx_n\rangle \xrightarrow[\infty]{n} 0,$$

because of the weak p-convergent property of T. We conclude from Lemma 3.1.17 that $\|Tx_n\| \xrightarrow[\infty]{n} 0$. □

The reader is referred to [3], page 14, for the definition of Dedekind σ-completeness of a vector lattice. Replacing the condition that F be a dual Banach lattice in Theorem 3.1.24 with the condition that F is Dedekind σ-complete, we obtain the following result:

Theorem 3.1.25 *Let E and F be two Banach lattices with F Dedekind σ-complete. If each positive weak p-convergent operator $T : E \to F$ is disjoint p-convergent, then one of the following assertions is valid:*

(1) E has the SP_p^+.

(2) F has an order continuous norm.

Proof Noting that by Theorem 4.51 in [3], the Dedekind σ-complete Banach lattice F does not have an order continuous norm if, and only, if ℓ_∞ is lattice embeddable in F, the proof follows as in $(1) \Rightarrow (2)$ of Theorem 3.1.24. □

As a consequence of Theorem 3.1.25, we obtain

Corollary 3.1.26 *Let E and F be two Banach lattices with F Dedekind σ-complete. If the norm of F is not order continuous, then the following assertions are equivalent:*

(1) Each positive operator $T : E \to F$ is disjoint p-convergent.

(2) Each positive weak p-convergent operator $T : E \to F$ is disjoint p-convergent.

(3) E has the SP_p^+.

Proof (1) $\Rightarrow$ (2): Trivial.
(2) $\Rightarrow$ (3): By Theorem 3.1.25.
(3) $\Rightarrow$ (1): Let $T : E \to F$ be any positive operator and let $(x_n) \in \ell_p^{weak}(E)$ be a disjoint sequence in E^+. By Proposition 3.1.21, $\|x_n\| \xrightarrow[\infty]{n} 0$. Since T is bounded, this implies that $\|Tx_n\| \xrightarrow[\infty]{n} 0$. Thus, by Lemma 5.1.4, T is disjoint p-convergent. □

3.2 Positive weak p-convergent and p-convergent operators

Throughout this section we assume that $1 \leq p < \infty$, unless otherwise specified. In some results of this section we need that all sequences $(x_n) \in \ell_p^{weak}(E)$ in a Banach lattice E satisfy $|x_n| \to 0$ weakly. Of course, this will be the case in Banach lattices for which the lattice operations are weakly sequentially continuous. It is well-known that the lattice operations in AM-spaces are weakly sequentially continuous. However, in the spaces $L_p[0,1]$ (where $1 \leq p < \infty$) the lattice operations fail to be weakly sequentially continuous (see the example on page 114 of [50]). The case $L_1[0,1]$ is also discussed in the solution to Problem 3.1.9 on page 91 of the book [1]. On the other hand, in Proposition 2.5.23 of [50] it is proved that in every atomic Banach lattice with order continuous norm the lattice operations are weakly sequentially continuous. The sequence spaces c_0 and ℓ_p ($1 \leq p < \infty$) are examples of atomic Banach lattices with order continuous norms, whereas the norm of c fails to be order continuous.

However, since we need the lattice operations to satisfy a seemingly weaker property than being weakly sequentially continuous, we introduce the notion "weakly sequentially p-continuous" as follows:

Definition 3.2.1 *The lattice operations in a Banach lattice E are said to be weakly sequentially p-continuous if the sequence $(|x_n|)$ converges weakly to 0 for every weakly p-summable sequence (x_n).*

Definition 3.2.2 *A Banach space X is said to have the Schur property of order p (briefly, X has the SP_p) if every weakly p-summable sequence is norm convergent to 0.*

It follows from the literature (cf. for instance [12], Proposition 2.1) that every weakly p-summable sequence in a Banach space X is norm convergent to 0 (for $1 \leq p < \infty$) if and only if $\ell_p^{weak}(X) = \ell_p^u(X)$. As is mentioned in the Preliminaries, it is a well-known fact that $\ell_1^{weak}(X) = \ell_1^u(X)$ if and only if X contains no copy of c_0. Thus, we immediately conclude that:

Proposition 3.2.3 *Let X be a Banach space which contains no copy of c_0. Then X has the SP_1.*

Corollary 3.2.4 *In each Banach lattice E which contains no copy of c_0, the lattice operations are weakly sequentially 1-convergent.*

Considering our discussion in the Preliminaries as well as the discussion in the paragraph before Proposition 3.2.3, it follows that a Banach space X has the SP_p (for $1 < p < \infty$) if and only if each bounded linear operator from $\ell^{p'}$ to X is compact. Similarly, a Banach space X has the SP_1 if each bounded linear operator from c_0 to X is compact. These observations provide us with an abundance of examples of Banach spaces which have the SP_p for some $1 \leq p < \infty$, but which do not have the Schur property:

Example 3.2.5 *(i) Suppose $1 < p < \infty$ and $1 < q < p'$ (i.e. $\frac{1}{q} + \frac{1}{p} > 1$), then by Pitt's Theorem (cf. for instance [2], Theorem 2.1.4) each bounded linear operator $T : \ell_{p'} \to \ell_q$ is compact. Thus, all the spaces ℓ_q have SP_p. In fact, more is true. Using the version of Pitt's Theorem as is discussed in [2] (on page 32), it follows that all closed subspaces of ℓ_q (for all $q < p'$) have SP_p. Of course, none of the spaces ℓ_q have the Schur property.*
(ii) Since it is also well-known that all bounded linear operators from c_0 into ℓ_p (for all $1 \leq p < \infty$) are compact (cf. for instance [2], Remark 2.1.5(b) on page 32) (or since ℓ_p does not contain a copy of c_0), it follows that all the ℓ_p-spaces have the SP_1. Again, none of the ℓ_p-spaces (for $p > 1$) have the Schur property.

On the other hand, we have the following example of a space with the DPP which does not have SP_2:

Example 3.2.6 *Let* (Ω, Σ, μ) *be some probability space. The space* $L_1(\mu)$ *has the* DPP *(by the Dunford-Pettis Theorem) and thus also has the* DPP_p *for all* $1 \leq p \leq \infty$*. By the above discussion, every weakly* 2*-summable sequence in* $L_1(\mu)$ *would be a norm null sequence if and only if each bounded linear operator from the sequence space* ℓ_2 *to* $L_1(\mu)$ *were compact. This is impossible, since for instance we know from Proposition 6.4.13 in [2] (page 155) that* ℓ_2 *embeds isometrically in* $L_1(\mu)$*. Thus, there has to be a weakly* 2*-summable sequence which is not norm null, showing that* $L_1(\mu)$ *does not have the* SP_2*.*

More examples of L_p-spaces without the Schur property of order p for some choices of p follow from Theorem 6.4.19 in [2]:

(i) For $1 \leq r \leq 2$, $\ell_{p'}$ embeds in L_r if and only if $r \leq p' \leq 2$. Thus, L_r does not have SP_p for all $2 \leq p \leq r'$.

(ii) For $2 < r < \infty$, $\ell_{p'}$ embeds in L_r if and only if $p' = 2$ or $p' = r$. Thus, L_r does not have SP_p for $p = 2$ or $p = r'$.

In general, the weak sequential continuity of the lattice operations in a Banach lattice is not implied by the weakly sequentially p-continuity of the same, as is illustrated by the following result:

Proposition 3.2.7 *The space* $L_1[0,1]$ *has* SP_1*. Thus, the lattice operations in* $L_1[0,1]$ *are weakly sequentially 1-continuous, but they are not weakly sequentially continuous.*

Proof By Corollary 5.2.11 in [2] (page 112), $L_1[0,1]$ does not contain a copy of c_0; i.e. by Proposition 3.2.3, $L_1[0,1]$ has SP_1. Thus, the lattice operations in $L_1[0,1]$ are weakly sequentially 1-continuous. It is however well-known that that the lattice operations in $L_1[0,1]$ are not weakly sequentially continuous (see the discussion above). □

Actually, more is known:

Example 3.2.8 *(i) Let* $1 \leq p < \infty$*. Recalling (from Corollary 10.7 in [30], page 200) that every weak* ℓ_1*-sequence in an* L_p*-space is a strong* ℓ_r *sequence where* $r = \max\{p, 2\}$*, it follows that any* L_p*-space has* SP_1*. Thus, the lattice*

operations in $L_p[0,1]$ are weakly sequentially 1-continuous, but they are not weakly sequentially continuous.
(ii) In any Banach space X which has cotype q (where $2 \leq q < \infty$) every weak ℓ_1 sequence is a strong ℓ_q sequence (cf. [30], Corollary 11.17, page 224). Thus, all Banach spaces with finite cotype, have SP_1.

We conclude this section by stating a result in [52] in our context of p-convergent operators. The reader should note that our arguments are similar to arguments in the proof of Theorem 2.4 of [52].

Theorem 3.2.9 *Let E and F be two Banach lattices. Then each positive weak p-convergent operator from E into F is p-convergent if one of the following assertions is valid:*

(1) F is a dual KB-space and the lattice operations in E are weakly sequentially p-continuous.

(2) F is a discrete KB-space.

(3) The norm of the topological bi-dual F^{**} is order continuous and the lattice operations in E are weakly sequentially p-continuous.

(4) E has the SP_p.

(5) F is reflexive.

Proof Let $T : E \to F$ be a positive weak p-convergent operator and let (x_n) be a weakly p-summable sequence in F.

(1) Let $T : E \to F$ be a positive weak p-convergent operator and let (x_n) be a weakly p-summable sequence. By our assumption $|x_n| \overset{n}{\underset{\infty}{\longrightarrow}} 0$ weakly in E. Since T is bounded, thus weakly continuous, it follows that $T(|x_n|) \overset{n}{\underset{\infty}{\longrightarrow}} 0$ weakly in F. This implies that $|Tx_n| \overset{n}{\underset{\infty}{\longrightarrow}} 0$ weakly in F, since $0 \leq |Tx_n| \leq T|x_n|$ for all n. Let $F = F_0^*$, then it follows for each $y \in F_0$ that $\langle |Tx_n|, y\rangle \overset{n}{\underset{\infty}{\longrightarrow}} 0$, since $y \in F_0^{**} = F^*$. Now let $g_n = Tx_n \in F_0^*$. In order to show that $\|Tx_n\|_F \overset{n}{\underset{\infty}{\longrightarrow}} 0$, by Lemma 3.1.17 we have to show that $|g_n| \overset{n}{\underset{\infty}{\longrightarrow}} 0$ weak* (i.e. $\sigma(F_0^*, F_0)$) and $g_n(z_n) \overset{n}{\underset{\infty}{\longrightarrow}} 0$ for every bounded disjoint sequence (z_n) in F_0^+. Let such

a sequence (z_n) be given. By the assumptions on $F = F_0^*$ (i.e. that F is a dual KB-space) it follows that $z_n \xrightarrow[\infty]{n} 0$ weakly in F_0, by Theorem 4.69 (cf. [3], page 245). Since the canonical injection $\tau : F_0 \to F_0^{**}$ is weakly continuous, $\tau(z_n) \xrightarrow[\infty]{n} 0$ weakly, i.e. $(\tau(z_n))_n$ is $\sigma(F_0^{**}, F_0^{***}) = \sigma(F^*, F^{**})$-convergent to 0. Since T is weak p-convergent, we have $\tau(z_n)(Tx_n) \xrightarrow[\infty]{n} 0$. However, $g_n(z_n) = \tau(z_n)(g_n) = \tau(Z_n)(Tx_n) \xrightarrow[\infty]{n} 0$. Also, $|g_n| = |Tx_n| \xrightarrow[\infty]{n} 0$ in the $\sigma(F, F^*)$-topology, i.e. $|g_n| \xrightarrow[\infty]{n} 0$ in the $\sigma(F_0^*, F_0^{**})$-topology, i.e. $|g_n| \xrightarrow[\infty]{n} 0$ weak*. Condition (2) in Lemma 3.1.17 is satisfied, hence $\|g_n\| \to 0$, i.e. $\|Tx_n\| \xrightarrow[\infty]{n} 0$.

(2) Since F is a discrete KB-space, it follows from Corollary 2.3 of [23], page 189 that the lattice operations are weakly sequentially continuous (hence weakly sequentially p-continuous). Therefore, if we assume that $(x_n) \in \ell_p^{weak}(E)$, then since $x_n \to 0$ weakly and T is bounded, we have $Tx_n \xrightarrow[\infty]{n} 0$ weakly in F. Therefore, $|Tx_n| \to 0$ weakly in F. Also, since each discrete KB-space is a dual Banach lattice (cf. [50]), we may now follow the arguments in the proof of (1) above to obtain that $\|Tx_n\| \xrightarrow[\infty]{n} 0$.

(3) Let $(x_n) \in \ell_p^{weak}(E)$. By assumption, $|x_n| \to 0$ weakly, i.e. as in the proof of (1) above, we have $|Tx_n| \xrightarrow[\infty]{n} 0$ weakly in F. To prove that $\|Tx_n\| \xrightarrow[\infty]{n} 0$, we make use of Lemma 3.1.16. Therefore, let (f_n) be a bounded disjoint sequence in $(F^*)^+$. Again, by Theorem 4.69 in [3], $f_n \to 0$ in the topology $\sigma(F^*, F^{**})$ (since F^{**} has order continuous norm). Since T is weak p-convergent, we obtain $f_n(Tx_n) \xrightarrow[\infty]{n} 0$. We have shown that assertion (2) of Lemma 3.1.16 is satisfied.
Thus $\|Tx_n\| \xrightarrow[\infty]{n} 0$.

(4) If we let $(x_n) \in \ell_p^{weak}(E)$, then $\|x_n\| \xrightarrow[\infty]{n} 0$, since E has the SP_p. Therefore, $\|Tx_n\| \to 0$ for each bounded linear operator $T : E \to F$.

(5) This follows from Corollary 3.1.13.

□

3.3 The Schur property of order p on Banach lattices

We will again assume throughout this section that $1 \leq p < \infty$, unless otherwise specified. The main result in the paper [62] (by Wickstead) is Theorem 2 (cf. [62], page 178), where the author considers necessary and sufficient conditions for the ideal property of positive Dunford-Pettis operators on Banach lattices to hold, i.e. to be able to conclude from $0 \leq S \leq T$ that S is Dunford-Pettis if T is. The following theorem shows that similar necessary conditions hold when we replace the Dunford-Pettis operators by p-convergent operators. Here the (clever) proof of the necessity part of Theorem 2 in [62] can be copied to prove our version. Although we refrain from discussing the proof in detail, we still opt to give a sketch of the proof and refer the reader to [62] for the details.

Theorem 3.3.1 *Let E and F be Banach lattices. Suppose the p-convergent operators from E to F satisfy the following ideal property: "If $S, T : E \to F$, with $0 \leq S \leq T$ such that T is p-convergent, then likewise S is p-convergent". Then at least one of the following conditions has to hold:*

(a) F has order continuous norm.

(b) The lattice operations in E are weakly sequentially p-continuous.

Proof Suppose neither (a) nor (b) holds. Let $(x_n) \in \ell_p^{weak}(E)$ with $(|x_n|)$ not weakly convergent to 0. Thus there is $f \in E^*$ with $f(|x_n|) \nrightarrow 0$. By choosing a subsequence and replacing f by $-f$ if necessary, we may assume there exists an $\epsilon > 0$ such that $f(|x_n|) \geq \epsilon > 0$ for all $n \in \mathbb{N}$. Since the same will be true for $|f|$, we may also assume that $f \geq 0$. Following the same arguments as in the proof of Theorem 2, [62] (page 178), we may construct g_n, $g \in [-f, f]$, with $g_n \to g$ (weak*) and $g_n(x_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. Since F does not have an order continuous norm, we know (by Corollary 2-4-2 in [50]) that there is an order bounded disjoint positive sequence in F which does not converge to zero in norm. By extracting a subsequence, we obtain a disjoint positive sequence (y_n) in F, which is bounded away from 0 in norm, and which is bounded from above by some $y \in F^+$ (as in the proof of Theorem 1 in [62]). Then we may continue as in the proof of Theorem 2 in [62], to define two operators $S, T : E \to F$ by

$$\begin{aligned} S(x) &= \sum_{n=1}^{\infty}(g_n(x) - g(x))y_n \\ T(x) &= 2f(x)y. \end{aligned}$$

T is rank one, and therefore p-convergent, while S is not p-convergent since although $(x_n) \in \ell_p^{weak}(E)$, we have

$$\|Sx_n\| \geq |g_n(x_n) - g(x_n)| \|y_n\| \nrightarrow 0.$$

Then clearly $-T \leq S \leq T$ so that $0 \leq S+T \leq 2T$, where $2T$ is p-convergent, but $S+T$ is not p-convergent. Thus, the ideal property does not hold. $\square$

Next we consider some classical Banach lattices, starting with the following lemma:

Lemma 3.3.2 *Let Ω be a compact Hausdorff topological space and let $1 \leq p < \infty$. For a weak p-summable sequence (f_j) in $C(\Omega)$ we have:*

(1) $(f_j(t))_j \in \ell_p$ *for all* $t \in \Omega$.

(2) $\sup_{t\in\Omega} \|(f_j(t))_j\|_{\ell_p} < \infty$.

Proof For $t \in \Omega$, let $\phi_t : C(\Omega) \to \mathbb{K}$ be the linear functional defined by $\phi_t(f) = f(t)$. Then $\phi_t \in C(\Omega)^*$ and $\|\phi_t\| \leq 1$. By assumption we have

$$(f_j(t))_j = (\phi_t(f_j))_j \in \ell_p$$

for all $t \in \Omega$. This proves (1).
It follows from the proof of (1) above that

$$\left(\sum_{j=1}^{\infty} |f_j(t)|^p\right)^{1/p} = \left(\sum_{j=1}^{\infty} |\phi_t(f_j)|^p\right)^{1/p} \leq \sup_{\|\phi\|\leq 1} \left(\sum_{j=1}^{\infty} |\phi(f_j)|^p\right)^{1/p} < \infty\,,$$

for all $t \in \Omega$. This proves (2). $\square$

Recall from the Riesz Representation Theorem that the space $M(\Omega)$ of regular Borel complex measures on Ω is the dual space of $C(\Omega)$ and that for each $\mu \in M(\Omega)$ we have

$$\langle f, \mu \rangle = \int_{\Omega} f(t)\, d\mu(t)\,,$$

for all $f \in C(\Omega)$.

Lemma 3.3.3 *Let $1 \le p < \infty$. If (f_j) is a weak p-summable sequence in $C(\Omega)$ then so is the sequence $(|f_j|)$.*

Proof Let (f_j) be weak p-summable in $C(\Omega)$ and let $\mu \in M(\Omega)$. Then for each $(\gamma_i) \in \ell_{p'}$ (with $1/p + 1/p' = 1$) we have

$$\sum_{i=1}^{m} |\gamma_i||\langle |f_i|, \mu\rangle| = \int_\Omega \left(\sum_{i=1}^{m} |\gamma_i||f_i(t)| \right) d\mu(t)$$

$$\le \int_\Omega \left(\sum_{i=1}^{m} |\gamma_i|^{p'} \right)^{1/p'} \left(\sum_{i=1}^{m} |f_i(t)|^p \right)^{1/p} d\mu(t) \text{ (by Lemma 3.3.2(1))}$$

$$\le \|(\gamma_i)\|_{p'} \int_\Omega \left(\sum_{i=1}^{\infty} |f_i(t)|^p \right)^{1/p} d\mu(t)$$

$$\le \|(\gamma_i)\|_{p'} \sup_{t\in\Omega} \left(\sum_{i=1}^{\infty} |f_i(t)|^p \right)^{1/p} \mu(\Omega) \text{ (by Lemma 3.3.2(2))},$$

for all $m \in \mathbb{N}$. Since (γ_i) was arbitrary, it follows that $(\langle \mu, |f_i|\rangle) \in \ell_p$ for all $\mu \in M(\Omega)$, i.e. that $(|f_i|)$ is weak p-summable in $C(\Omega)$. □

For any AM-space with unit, being lattice isometric to some $C(\Omega)$-space (cf. Theorem 4.29 of [3]), it follows from the above lemmas that:

Proposition 3.3.4 *Let E be an AM-space with unit. Then $(|x_i|) \in \ell_p^{weak}(E)$ for each $(x_i) \in \ell_p^{weak}(E)$.*

We introduce the following notion:

Definition 3.3.5 *We say a Banach lattice E is weak p-consistent (for $1 \le p < \infty$) if it follows from $(x_i) \in \ell_p^{weak}(E)$ that $(|x_i|) \in \ell_p^{weak}(E)$.*

It is clear that if a Banach lattice E has the SP_p, then the lattice operations are in particular weakly sequentially p-continuous and E has the SP_p^+. On the other hand, if E is a weak p-consistent Banach lattice (for instance an AM-space with unit) and E has the SP_p^+, then for each $(x_n) \in \ell_p^{weak}(E)$ we have $(|x_n|) \in \ell_p^{weak}(E)$ and so $\|x_n\| = \||x_n|\| \to 0$ as $n \to \infty$. Thus, we have:

Proposition 3.3.6 *Let E be a weak p-consistent Banach lattice. Then the following are equivalent:*

(i) E has the SP_p.

(ii) E has the SP_p^+

The following result is a partial converse of Theorem 3.3.1, assuming a stronger property than in Theorem 3.3.1(b):

Proposition 3.3.7 *Let E be a weak p-consistent Banach lattice and F any Banach lattice. If $S, T : E \to F$ are positive operators satisfying $0 \leq S \leq T$ and T is p-convergent, then likewise S is p-convergent.*

Proof Since for each $(x_i) \in \ell_p^{weak}(E)$, we also have $(|x_i|) \in \ell_p^{weak}(E)$, then $\| T|x_n| \| \to 0$ as $n \to \infty$. Thus we have

$$\|Sx_n\| = \| |Sx_n| \| \leq \|S|x_n| \| \leq \| T|x_n| \| \to 0$$

as $n \to \infty$. □

When the target space is an AL-space, then we have the following easy characterisation of a p-convergent operator:

Proposition 3.3.8 *Let E be a Banach lattice and let F be an AL-space. Then the following are equivalent:*

(1) T is p-convergent.

(2) $|Tx_n| \to 0$ as $n \to \infty$ weakly in F for all $(x_i) \in \ell_p^{weak}(E)$.

Proof (1) $\implies$ (2) is clear from $\| |Tx_n| \| = \|Tx_n\| \to 0$ as $n \to \infty$.
To prove (2) $\implies$ (1), observe that the linear functional $e \in F^*$ defined by

$$e(y) := \|y^+\| - \|y^-\|$$

on the AL-space F satisfies $e(|y|) = \|y\|$ for all $y \in F$ (see [3], page 200). Thus,

$$\|Tx_n\| = e(|Tx_n|) \to 0$$

for all $(x_i) \in \ell_p^{weak}(E)$; i.e. T is a p-convergent operator. □

From the previous result we see that:

Proposition 3.3.9 *Let E be a Banach lattice and let F be an AL-space in which the lattice operations are weakly sequentially p-continuous, then each positive operator $T : E \to F$ is p-convergent.*

Proof Being positive, T is bounded. Thus, if $(x_i) \in \ell_p^{weak}(E)$ is given, then $(Tx_i) \in \ell_p^{weak}(F)$. By assumption, $|Tx_n| \to 0$ weakly. Therefore, by Proposition 3.3.8, the operator T is p-convergent. □

Remark 3.3.10 *Let E be a Banach lattice. From Proposition 3.2.7 and Proposition 3.3.9 it follows that each positive operator $T : E \to L_1[0,1]$ is 1-convergent.*

Using Theorem 3.1.22, we obtain a similar characterization for the SP_p on a Banach lattice:

Theorem 3.3.11 *Let E be a Banach lattice. Then, the following assertions are equivalent:*

(1) Each positive operator from E into ℓ_∞ is p-convergent.

(2) E has the SP_p.

Proof (1) $\Longrightarrow$ (2) : It follows from Theorem 3.1.22 that E has SP_p^+. On the other hand, since each positive operator from E into ℓ_∞ is p-convergent and the norm of ℓ_∞ is not order continuous, then it follows that the lattice operations in E are weakly sequentially p-continuous (by Theorem 3.3.1). Now if (x_n) is a weakly p-summable sequence, then $|x_n| \to 0$ weakly in E. Since E has the SP_p^+, it follows that $\|x_n\| = \||x_n|\| \to 0$. Thus E has the SP_p.
(2) $\Longrightarrow$ (1) : This is trivial. □

Theorem 3.3.12 *Let E and F be two Banach lattices with F Dedekind σ-complete. If each positive weak p-convergent operator from E into F is p-convergent, then one of the following assertions is valid:*

(1) E has the SP_p.

(2) F has an order continuous norm.

Proof It is enough to establish that if the norm of F is not order continuous, then E has the SP_p. So, suppose that the norm of F is not order continuous and consider an arbitrary positive operator $T : E \to \ell_\infty$. Since ℓ_∞ is lattice embeddable in F ([3], Theorem 4.51, p. 227) and ℓ_∞ has DPP_p, then T is a weak p-convergent operator (Remark 3.1.15 (c)). Let $J : \ell_\infty \hookrightarrow F$ denote the lattice embedding. By our assumption, $J \circ T$ is p-convergent and hence T is p-convergent (cf. Proposition 3.1.3). Finally, Theorem 3.3.11 finishes the proof. □

The assumption that F is Dedekind σ-complete in the above theorem is because we refer to [3] (Theorem 4.51) where the same assumption was made. This is not essential if the space E is separable, as will be illustrated by the discussion (in Theorem 3.3.14) below.

Recall from [37] that a Banach space X is said to have the DP^*-property of order p (DP^*P_p for short) if for $(x_n) \in \ell_p^{weak}(X)$ and $(x_n^*) \in c_0^{weak^*}(X^*)$ we have $\langle x_n^*, x_n \rangle \to 0$ as $n \to \infty$. By Proposition 2.8 in [37], X has DP^*P_p if and only if each bounded linear operator $T : X \to c_0$ is p-convergent.

Lemma 3.3.13 *A separable Banach space X with the DP^*P_p has the Schur property of order p.*

Proof Suppose X has DP^*P_p, but not the SP_p. Then there exists $(x_i) \in \ell_p^{weak}(X)$ such that $\|x_n\| \nrightarrow 0$. Taking subsequences if necessary, we may assume that $\|x_n\| \geq \epsilon$ for some fixed $\epsilon > 0$. Let $x_n^* \in X^*$ such that $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\| \geq \epsilon$ for all n. The unit ball B_{X^*}, being weak* sequentially compact (because X is separable), there has to be a subsequence $(x_{n_k}^*)_k$ such that $x_{n_k}^* \to x^* \in B_{X^*}$ weak*. Using that X has DP^*P_p, it follows that

$$\langle x_{n_k}^* - x^*, x_{n_k} \rangle \to 0 \text{ as } k \to \infty\ .$$

Since $\langle x^*, x_{n_k} \rangle \to 0$, this contradicts the assumption that

$$x_{n_k}^*(x_{n_k}) = \|x_{n_k}\| \geq \epsilon$$

for all n_k. □

Theorem 3.3.14 *Let E and F be two Banach lattices such that E is separable. If each weak p-convergent operator $T : E \to F$ is p-convergent, then one of the following assertions is valid:*

(1) E has the SP_p.

(2) F is KB-space.

Proof Suppose that F is not a KB-space and consider an arbitrary bounded linear operator $T : E \to c_0$. Since c_0 is lattice embeddable in F (Theorem 4.60, [3]) and c_0 has DPP_p, then T is weak p-convergent (cf. Remark 3.1.15), and hence it is p-convergent by our assumption, i.e. each bounded linear operator $T : E \to c_0$ is p-convergent. This shows that E has the DP^*P_p (by [37], Proposition 3.6). By Lemma 3.3.13 the space E has the SP_p. □

Corollary 3.3.15 *Let E and F be two Banach lattices such that E is separable and F is not a KB-space. Then the following assertions are equivalent:*

(1) Each bounded linear operator $T : E \to F$ is p-convergent.

(2) E has the SP_p.

Proof The implication (2) $\Longrightarrow$ (1) is clear.
(1) $\Longrightarrow$ (2) : This follows from Theorem 3.3.14, since then each weak p-convergent operator $T : E \to F$ is p-convergent. □

Recall that a Banach space X has the Gelfand-Phillips property if all limited subsets of X are relatively compact, i.e. a subset A of X is relatively compact if and only if every weak*-null sequence in X^* converges uniformly on A (cf. for instance [33], page 405). Separable Banach spaces satisfy the Gelfand-Phillips property. It is therefore natural to consider Theorem 3.3.14 for Banach lattices with the Gelfand-Phillips property. Before doing so, we need to consider some preliminary results for Banach spaces with this property. It is remarked in the paper [33] that:

(I) A sequence (x_n) in X is limited (i.e. the set $\{x_n : n \in \mathbb{N}\}$ is limited) if and only if $x_n^*(x_n) \xrightarrow{n} 0$ for all weak*-null sequences (x_n^*) in X^*.
(II) X is a Gelfand-Phillips space if and only if every limited weakly null sequence in X is norm null.

Lemma 3.3.16 *Let X be a Banach space. If each bounded linear operator $T : X \to c_0$ is p-convergent, then each sequence $(x_n) \in \ell_p^{weak}(X)$ is limited.*

Proof By the assumption that each bounded linear operator $T : X \to c_0$ is p-convergent, it follows that X has the DP^*P_p. Thus, if $(x_n) \in \ell_p^{weak}(X)$ and $(x_n^*) \in c_0^{weak^*}(X^*)$, then $\langle x_n^*, x_n \rangle \xrightarrow{n} 0$. By (I) above, the sequence (x_n) is limited. □

Lemma 3.3.17 *Let X be a Gelfand-Phillips space for which each bounded linear operator $T : X \to c_0$ is p-convergent. Then each $(x_i) \in \ell_p^{weak}(X)$ is norm null.*

Proof This follows from Lemma 3.3.16 and (II) above. □

From the above lemmas we obtain the following extension of Theorem 3.3.14:

Theorem 3.3.18 *Let E and F be two Banach lattices such that E is a Gelfand-Phillips space. If each weak p-convergent operator $T : E \to F$ is p-convergent, then one of the following assertions is valid:*

(1) E has the SP_p.

(2) F is KB-space.

Proof As in the proof of Theorem 3.3.14 we assume that F is not a KB-space and conclude that each bounded linear operator $T : E \to c_0$ is p-convergent. Since E is a Gelfand-Phillips space, each $(x_i) \in \ell_p^{weak}(X)$ is norm null by Lemma 3.3.17. Thus, E has the SP_p. □

It is known that Banach lattices with order continuous norms have the Gelfand-Phillips property (as is remarked in [64]). Therefore, it follows from Theorem 3.3.18 that:

Corollary 3.3.19 *Let E and F be two Banach lattices such that E has an order continuous norm and F is not a KB-space. Then the following assertions are equivalent:*

(1) Each operator $T : E \to F$ is p-convergent.

(2) E has the SP_p.

Chapter 4

On weak* p-convergent operators

The purpose of this chapter is to introduce and study the notion of "weak* p-convergent operator". We discuss the relationship between the weak* p-convergent operators and the p-convergent operators, a class of operators that was mentioned in Chapter 3 and which plays an important role in the study of the DP^*-property of order p (introduced in [69]). Some new characterizations of Banach spaces with the DP^*-property of order p are obtained, the p-Gelfand-Phillips property is introduced and the behaviour of weak* p-convergent operators on Banach spaces with this property (with focus on Banach lattices with the p-Gelfand-Phillips property) is investigated. In Section 4.2, we consider the domination properties of positive p-convergent and positive weak* p-convergent operators on Banach lattices. Please note that the main results in this chapter was submitted and accepted for publication (cf. [67]).

4.1 Weak* p-convergent operators

Throughout this chapter we assume that $1 \leq p < \infty$, unless otherwise stated. The concept "weak p-convergent operator" was introduced in Chapter 3. We recall that a bounded linear operator $T : X \to Y$ is said to be weak p-convergent if $y_n^*(Tx_n) \xrightarrow[\infty]{n} 0$ for every $(x_n) \in \ell_p^{weak}(X)$ and every $(y_n^*) \in c_0^{weak}(Y^*)$. Since we know that in general $c_0^{weak}(Y^*) \subset c_0^{weak^*}(Y^*)$ and that the inclusion may be strict, it makes sense to introduce the following operator:

Definition 4.1.1 *An operator T from a Banach space X into a Banach space Y is called weak* p-convergent if $(y_n^*(Tx_n))$ converges to 0 for every $(x_n) \in \ell_p^{weak}(X)$ and every $(y_n^*) \in c_0^{weak^*}(Y^*)$.*

It follows easily from the definition that:

Proposition 4.1.2 *If $S : X \to Y$ is weak* p-convergent, $T \in \mathcal{L}(X_0, X)$, $R \in \mathcal{L}(Y, Y_0)$, then $RST : X_0 \to Y_0$ is weak* p-convergent.*

Clearly each weak* p-convergent operator is weak p-convergent. Recall from Chapter 3 that an operator T from a Banach space X into a Banach space Y is said to be p-convergent if the sequence $(\|Tx_n\|)$ converges to 0 in Y, for every sequence $(x_n) \in \ell_p^{weak}(X)$. Since weak* null sequences are norm bounded, it is clear that p-convergent operators are weak* p-convergent. In the following proposition another connection between the weak* p-convergent and p-convergent operators is discussed:

Proposition 4.1.3 *Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. The following are equivalent:*

(a) T is weak p-convergent.*

(b) ST is p-convergent for each $S \in \mathcal{L}(Y, Z)$ and any separable Banach space Z.

(c) ST is p-convergent for each $S \in \mathcal{L}(Y, c_0)$.

Proof Clearly, $(b) \implies (c)$. We prove:
$(a) \implies (b)$: Assume T is weak* p-convergent. Suppose ST is not p-convergent for some $S \in \mathcal{L}(Y, Z)$ and some separable Banach space Z. Then there exists $(x_n) \in \ell_p^{weak}(X)$ such that $\|STx_n\| \nrightarrow 0$ as $n \to \infty$. Taking subsequences we may therefore assume that there exist an $\varepsilon > 0$ and a sequence $(y_n^*) \subset \mathcal{B}_{Z^*}$ such that $|\langle y_n^*, STx_n\rangle| \geq \varepsilon$ for all $n \in \mathbb{N}$. Since Z is separable and $\mathcal{B}_{Z^*}$ is weak*-compact, there exists a subsequence $(y_{n_k}^*)$ such that $y_{n_k}^* \to y^* \in \mathcal{B}_{Z^*}$ weak*. Since $S^* : Z^* \to Y^*$ is weak*-weak* continuous ([24], Proposition VI.1.3, page 167), we have $S^*y_{n_k}^* \to S^*y^*$ weak*. Put $h_{n_k}^* = S^*y_{n_k}^* - S^*y^*$, i.e. $h_{n_k}^* \to 0$ weak* in F^*. By assumption, $\langle h_{n_k}^*, Tx_{n_k}\rangle \xrightarrow[\infty]{k} 0$.

Also, since $STx_{n_k} \to 0$ weakly, because ST is weak-weak continuous (cf. [24], Theorem VI.1.1, page 166), we have

$$\begin{aligned}\langle y^*_{n_k}, STx_{n_k}\rangle &= \langle h^*_{n_k}, Tx_{n_k}\rangle + \langle y^*, STx_{n_k}\rangle \\ &\xrightarrow[\infty]{k} 0.\end{aligned}$$

This contradicts the assumption $|\langle y^*_{n_k}, STx_{n_k}\rangle| \geq \varepsilon$ for all $k \in \mathbb{N}$.
$(c) \implies (a)$: Suppose for each bounded linear operator S from Y into c_0, the operator ST is p-convergent. Let $(x_n) \in \ell_p^{weak}(X)$ and $(y^*_n) \in c_0^{weak^*}(Y^*)$. If we put $S = L_{(y^*_i)}$, then $S \in \mathcal{L}(Y, c_0)$ and by assumption ST is p-convergent, thus weak* p-convergent. Thus, since $(e_n) \subset c_0^* = \ell_1$ is weak* convergent to 0, it follows that $\langle e_n, STx_n\rangle \xrightarrow[\infty]{n} 0$. However,

$$\begin{aligned}\langle e_n, STx_n\rangle &= \langle e_n, S(Tx_n)\rangle \\ &= \langle e_n, (\langle Tx_n, y^*_i\rangle)_i\rangle \\ &= \langle y^*_n, Tx_n\rangle.\end{aligned}$$

This proves that T is weak* p-convergent. □

It follows from Proposition 4.1.3 that

Corollary 4.1.4 *If X, Y are Banach spaces, with Y separable, then each weak* p-convergent operator $T : X \to Y$ is p-convergent.*

Comparing this result with Corollary 3.1.13, we have:

Proposition 4.1.5 *If X, Y are Banach spaces, with Y a separable reflexive space, then the families of p-convergent operators, weak p-convergent operators and weak* p-convergent operators from X to Y coincide.*

The concept of a limited set (see Chapter 3 for definition) has been widely studied in the literature. In [29] (page 116) several properties of limited sets are listed: Limited sets are norm bounded and relatively compact sets are limited; in separable Banach spaces, however, limited sets are relatively compact; the set $\{e_n : n \in \mathbb{N}\}$ of unit vectors is limited in ℓ_∞, but not in c_0. An operator T from a Banach space X into another Banach space Y is said to be a *limited operator* if it carries the closed unit ball of X into a limited set of Y.

In Chapter 3, the concept of relatively weakly p-compact (respectively, weakly p-compact) was mentioned. The authors in [20] call an operator $T : X \to Y$ *weakly p-compact* if $T(B_X)$ is relatively weakly p-compact in Y. The family $\mathcal{W}_p$ of all weakly p-compact operators on the family of all Banach spaces is an injective and surjective non-closed operator ideal (when normed by the uniform operator norm). Obviously, $\mathcal{W}_\infty = \mathcal{W}$. It is important to note that if $1 < p < \infty$, then based upon the reflexiveness of $\ell_{p'}$ and the Bessaga-Pelczynski selection principle, it follows that $id_{\ell_{p'}} \in \mathcal{W}_p$ (cf. [20], Proposition 1.4). Therefore, if $1 < p < \infty$, then for all Banach spaces X and each $T \in \mathcal{L}(\ell_{p'}, X)$ the operator T is weakly p-compact. Thus, given any $(x_i) \in \ell_p^{weak}(X)$, then the set

$$\{\sum_{i=1}^{\infty} \lambda_i x_i : (\lambda_i) \in B_{\ell_{p'}}\} = R_{(x_i)}(B_{\ell_{p'}})$$

is relatively weakly p-compact. Any subset of $R_{(x_i)}(B_{\ell_{p'}})$ (for every $(x_i) \in \ell_p^{weak}(X)$) will therefore also be relatively weakly p-compact. By the discussion in [20], for a Banach space Z we have $id_Z \in \mathcal{W}_1$ if and only if Z is finite dimensional. Thus the case $p = 1$ is excluded from the above discussion.

There is some confusion in the modern literature concerning the concept of a "relatively weakly p-compact set" as was first defined in [20]. For instance, in the paper [60] (see Definition 2.3 in [60]) and several other recent papers, a subset K of a Banach space X is called a "relatively weakly p-compact set" if K is contained in $R_{(x_i)}(B_{\ell_{p'}})$ for some $(x_i) \in \ell_p^{weak}(X)$. By our discussion above (for $1 < p < \infty$), this is a stronger concept than the definition of "relatively weakly p-compact set" as was given in the paper [20]. A recent explanation by Bill Johnson to Dongyang Chen (which the authors found on the internet in the "mathoverflow" webpage) shows that the two concepts discussed in the papers [20] and [60] (under the same name) are indeed not in general the same: Let, for instance,

$$X = \left(\sum_{n=1}^{\infty} \ell_1^n\right)_{p'} \quad \text{with } 1 < p < \infty.$$

Then B_X is relatively weakly p-compact in the sense of [20] (i.e. each sequence in B_X has a weakly p-convergent subsequence). However, if we assume that $B_X \subseteq R_{(x_i)}(B_{\ell_{p'}})$ for some $(x_i) \in \ell_p^{weak}(X)$, then this implies that $R_{(x_i)}$ is a surjection (i.e. that X is isomorphic to a quotient space of $\ell_{p'}$). This cannot be true, since X^* is definitely not isomorphic to a subspace of $(\ell_{p'})^* = \ell_p$. We opt to use "relatively weakly p-compact set" as is defined in [20]. Recall that in Chapter 3 the concept of the DP^*P_p was mentioned.

Theorem 4.1.6 *(cf. [37], Theorem 2.4) Let $1 \leq p < \infty$. The Banach space X has DP^*P_p if and only if $\langle x_n^*, x_n \rangle \to 0$ as $n \to \infty$ for all weak* null sequences $(x_n^*) \subset X^*$ and all weakly p-summable sequences $(x_n) \subset X$.*

Corollary 4.1.7 *A Banach space X has DP^*P_p if and only if the identity operator id_X is weak* p-convergent. If X is separable, then by Corollary 4.1.4, this is equivalent to id_X being p-convergent.*

Remark 4.1.8 *From the above results we observe the following interesting facts:*
(a) In the paper [19] a stronger version of the Dunford-Pettis property, named DP^*-property (which will be the DP^*P_∞ in Definition 2.3 of paper [37], if we agree to replace "weakly ∞-compact sets" by "weakly compact sets") is introduced. Both ℓ_1 and ℓ_∞ have this property (cf. [19]). Since for $1 \leq p < \infty$, the DP^*-property implies the DP^*P_p-property, the space ℓ_∞ therefore also has the DP^*P_p. Thus each $T \in \mathcal{L}(X, \ell_\infty)$ is weak p-convergent for any Banach space X; in particular, the identity operator on ℓ_∞ is weak* p-convergent. However, id_{ℓ_∞} is not p-convergent (for $1 \leq p < \infty$): If it were p-convergent, then by our discussion in Section 1.2 it would mean that $\ell_p^u(\ell_\infty) = \ell_p^{weak}(\ell_\infty)$, i.e. that each bounded linear operator from $\ell_{p'}$ (respectively, c_0 if $p = 1$) into ℓ_∞ would be compact. In particular, the embedding $\ell_{p'} \hookrightarrow \ell_\infty$ would be compact. This is, of course, not true since for instance $(e_n) \subset \mathcal{B}_{\ell_{p'}}$ and for each subsequence (e_{n_k}) we have $\|e_{n_k} - e_{n_l}\|_{\ell_\infty} = 1$ for all $k \neq l$.*

(b) It is also proven in [37] that a Banach space X has DP^*P_p if and only if each bounded linear operator $T : X \to c_0$ is p-convergent. As in the discussion for ℓ_∞ above, id_{c_0} is not p-convergent. Thus, by Corollary 4.1.4, id_{c_0} is not weak p-convergent. Therefore, although c_0 has the DPP (and therefore also the DPP_p), it does not have the DP^*P_p.*

Proposition 4.1.9 *Let X be a Banach space. Then the following statements are equivalent:*

(a) Every relatively weakly p-compact set in X is limited.

(b) For every weakly p-summable sequence (x_n) in X, the set $\{x_n : n \in \mathbb{N}\}$ is limited in X.

(c) X has DP^*P_p.

Proof (a) $\Longrightarrow$ (b): Let $(x_n) \in \ell_p^{weak}(X)$ and consider the corresponding set $A := \{x_n : n \in \mathbb{N}\}$. Each sequence in A with finitely many different terms clearly has a subsequence (consisting of an element which appears infinitely many times in the sequence) which is weakly p-convergent to an element in A. On the other hand, each sequence in A with infinitely many different terms has a subsequence consisting of a permutation of a subsequence of (x_n) and therefore the subsequence belongs to $\ell_p^{weak}(X)$ again, i.e. each sequence in A has a weak p-convergent subsequence. This means that $\{x_n : n \in \mathbb{N}\}$ is relatively weakly p-compact and thus by our hypothesis, it is limited.

(b) $\Longrightarrow$ (c): Given arbitrary $(x_n) \in \ell_p^{weak}(X)$ and $(x_n^*) \in c_0^{weak^*}(X^*)$, the result follows from

$$|\langle x_n^*, x_n\rangle| \leq \sup_k |\langle x_n^*, x_k\rangle| \xrightarrow[\infty]{n} 0.$$

(c) $\Longrightarrow$ (a): Let $W \subset X$ be a relatively weakly p-compact set and let $(x_n^*) \in c_0^{weak^*}(X^*)$. We are required to prove that (x_n^*) converges uniformly on W. If not, then we may assume (by taking a subsequence if necessary and call it (x_n^*) again) that there exist $\varepsilon > 0$ and a sequence $(x_n) \subset W$ such that $\forall n \in \mathbb{N}$, $|\langle x_n^*, x_n\rangle| \geq \varepsilon$. Since W is relatively weakly p-compact, there exists a subsequence (x_{n_k}) of (x_n) and an $x \in X$ such that $(x_{n_k} - x) \in \ell_p^{weak}(X)$. By the hypothesis, $|\langle x_{n_k}^*, x_{n_k} - x\rangle| \to 0$ as $k \to \infty$. Thus we obtain

$$|\langle x_{n_k}^*, x_{n_k}\rangle| \leq |\langle x_{n_k}^*, x_{n_k} - x\rangle| + |\langle x_{n_k}^*, x\rangle| \to 0 \text{ if } k \to \infty.$$

This contradicts that $|\langle x_{n_k}^*, x_{n_k}\rangle| \geq \varepsilon$ for all k. Since $(x_n^*) \in c_0^{weak^*}(X^*)$ is arbitrary, it follows that W is a limited set in X. □

Theorem 4.1.10 *Let $1 < p < \infty$. Let X and Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. Then the following statements are equivalent:*

(a) T is weak p-convergent.*

(b) T carries relatively weakly p-compact subsets of X to limited subsets of Y.

(c) For any Banach space Z and each weakly p-compact operator S from Z to X, the operator $T \circ S$ is limited.

(d) For any bounded linear operator S from $\ell_{p'}$ into X, the operator $T \circ S$ is limited.

Proof $(a) \implies (b)$: Let T be weak* p-convergent and suppose there exists a relatively weakly p-compact subset W of X such that $T(W)$ fails to be limited. Let $(y_n^*) \in c_0^{weak^*}(Y^*)$ such that (y_n^*) does not converge uniformly (to 0) on $T(W)$. Then there exist $\epsilon > 0$ and a subsequence $(y_{n_k}^*)$ such that for each k there exists $x_k \in W$ so that $|y_{n_k}^*(Tx_k)| \geq \varepsilon$. W being relatively weakly p-compact, there is a subsequence (x_{k_j}) of (x_k) and an $x \in X$ such that $(x_{k_j} - x) \in \ell_p^{weak}(X)$. T being weak* p-convergent, we have

$$|\langle y_{n_{k_j}}^*, T(x_{k_j} - x)\rangle| \to 0 \text{ as } j \to \infty.$$

This implies that

$$|\langle y_{n_{k_j}}^*, Tx_{k_j}\rangle| \leq |\langle y_{n_{k_j}}^*, T(x_{k_j} - x)\rangle| + |\langle y_{n_{k_j}}^*, Tx\rangle| \xrightarrow{j} 0,$$

contradicting that $|y_{n_{k_j}}^*(Tx_{k_j})| \geq \varepsilon$ for all j.
$(b) \implies (c)$: Let Z be a Banach space and let $S : Z \to X$ be a weakly p-compact operator. Then $S(\mathcal{B}_Z)$ is a relatively weakly p-compact subset of X and our hypothesis implies that $T(S(\mathcal{B}_Z))$ is a limited set of Y, thus $T \circ S$ is a limited operator.
$(c) \implies (d)$: This is clear, because $\mathrm{id}_{\ell_{p'}}$ is weakly p-compact and so each $S \in \mathcal{L}(\ell_{p'}, X)$ is weakly p-compact.
$(d) \implies (a)$: Let $(x_n) \in \ell_p^{weak}(X)$ and $(y_n^*) \in c_0^{weak^*}(Y^*)$. Then $T \circ R_{(x_n)}$ is a limited operator and thus the set $(T \circ R_{(x_n)})(B_{\ell_{p'}})$ is limited. Furthermore,

$$\begin{aligned} |\langle y_n^*, Tx_n\rangle| &= |\langle y_n^*, T(R_{(x_n)}e_n)\rangle| \\ &\leq \sup_{(\lambda_k) \in \mathcal{B}_{\ell_{p'}}} |\langle y_n^*, (T \circ R_{(x_n)})((\lambda_k)_{k=1}^{\infty})\rangle| \xrightarrow[\infty]{n} 0. \end{aligned}$$

This shows that T is weak* p-convergent. □

As a consequence of Theorem 4.1.10 and Corollary 4.1.7, we get

Corollary 4.1.11 *Let $1 < p < \infty$ and let X be a Banach space. Then the following statements are equivalent:*

(a) X has DP^*P_p.

(b) The identity operator of X is weak p-convergent, i.e. every relatively weakly p-compact set of X is a limited set.*

(c) Every weakly p-compact operator T from an arbitrary Banach space Z to X is a limited operator.

(d) Every bounded linear operator $T : \ell_{p'} \to X$ is a limited operator.

We call a sequence (x_n) in a Banach space X *limited* if the corresponding set of all its terms is a limited set. If the same sequence is also a weakly null (respectively, a weakly p-summable) sequence, then we call (x_n) a limited weakly null sequence (respectively, a limited weakly p-summable sequence) in X. Recall that a Banach space X is said to have the Gelfand-Phillips property (GPP for short) or X is said to be a Gelfand-Phillips space, if all limited subsets of X are relatively (norm) compact. Refer to [33] for the following two facts:

(A) A sequence (x_n) in X is limited if and only if $x_n^*(x_n) \to 0$ for each weak* null sequence (x_n^*) in X^*.

(B) X is a Gelfand-Phillips space if and only if every limited weakly null sequence in X is norm null.

With (A) and (B) in mind, we introduce the following definition of the p-Gelfand-Phillips property:

Definition 4.1.12 *Let $1 \leq p < \infty$. A Banach space X is said to have the p-Gelfand-Phillips property (p-GPP for short) if every limited weakly p-summable sequence (x_n) in X is norm null. If X has this property, then we call X a p-Gelfand-Phillips space.*

The case of $p = \infty$ could have been included in Definition 4.1.12 if we would consider the ∞-Gelfand-Phillips property the same as the GPP. Clearly, if $1 \leq p < q$, then each limited weakly p-summable sequence in a Banach space X will also be a limited weakly q-summable sequence and so the q-GPP will imply the p-GPP on X and they will be implied by the GPP. Some classical Banach spaces, such as c_0 and ℓ_1 have the GPP and thus they also have the p-GPP for all $1 \leq p < \infty$. It is known that ℓ_∞ lacks the GPP. Using that each $\phi \in (\ell_\infty)^*$ can be written as $\phi = (\xi_i) + f$ such that $(\xi_i) \in \ell_1$ and $f \in c_0^\perp$ (see for instance [34], page 305; this also follows from a general discussion in [39]) one easily verifies that the sequence $(e_n)_{n\in\mathbb{N}}$ is a limited weakly p-summable sequence in ℓ_∞ for all $1 \leq p < \infty$. Thus, ℓ_∞ does not have the p-GPP for any $1 \leq p < \infty$.

Motivated by a definition in [46], we define

Definition 4.1.13 *A bounded linear operator $T : X \to Y$ between two Banach spaces is called limited p-convergent if it carries limited weakly p-summable sequences in X to norm null ones in Y.*

Clearly, a Banach space X has the p-GPP if and only if id_X is limited p-convergent. By definition, p-convergent operators are limited p-convergent.

By Corollary 4.1.4, if Y is separable, then weak* p-convergent operators with target space Y are limited p-convergent. In particular, the identity operators on separable spaces with the DP^*P_p are limited p-convergent, i.e. separable spaces with the DP^*P_p are p-Gelfand-Phillips spaces. From our discussions above we know that for all $1 \leq p < \infty$ the non-separable space ℓ_∞ has the DP^*P_p, but it does not have the p-GPP.

In Chapter 2 we considered Banach lattices E and F for which weak* Dunford-Pettis operators from E to F are limited completely continuous and found conditions on the underlying Banach lattices that ensured that weak* p-convergent operators are limited p-convergent.

In [3] on page 227, the authors state that for an arbitrary Dedekind σ-complete Banach lattice E, ℓ_∞ is lattice embeddable in E if and only if E does not have order continuous norm. We use this statement in the following theorem:

Theorem 4.1.14 *(compare with Theorem 2.4.9) Let E and F be Banach lattices such that F is Dedekind σ-complete. Then the following assertions are equivalent:*

(1) Each weak p-convergent operator T from E into F is limited p-convergent.*

(2) Either E has the p-GPP or F has an order continuous norm.

Proof (1) $\implies$ (2): Assume that E does not have the p-GPP and that the norm of F is not order continuous. Then there exists a limited weakly

p-summable sequence (x_n) in E satisfying $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Choose a sequence $(f_n) \subseteq E^*$ with $\|f_n\| = 1$ such that $\frac{1}{2} \leq |f_n(x_n)| \leq 1$ for all n and consider the norm ≤ 1 bounded linear operator

$$Q : E \to \ell_\infty : x \mapsto (f_n(x))_{n=1}^\infty.$$

By the assumptions on F and the statement in the paragraph preceding the theorem, there exists a lattice embedding $S : \ell_\infty \to F$. We may thus assume that there exists a positive constant K such that

$$K\|(\alpha_k)_{k=1}^\infty\|_{\ell_\infty} \leq \|S((\alpha_k)_{k=1}^\infty)\|_F$$

for all $(\alpha_i) \in \ell_\infty$. The operator $T = S \circ Q : E \to F$ is weak* p-convergent (since ℓ_∞ has DP^*P_p). It is, however, not limited p-convergent: Indeed, since the sequence (x_n) is limited and weakly p-summable in F, but

$$\|T(x_n)\| = \|S(f_n(x_n))_{n=1}^\infty\| \geq K\|(f_n(x_n))_{n=1}^\infty\|_{\ell_\infty} \geq K/2$$

for each $n \in \mathbb{N}$, it follows that T is not limited p-convergent.

(2) $\implies$ (1): First, suppose E has the p-GPP. Then each bounded linear operator $T : E \to F$ is limited p-convergent. On the other hand, if F is Dedekind σ-complete and has an order continuous norm, then by Theorem 4.5 in [66], F has the GP-property, hence the p-GPP and again each bounded linear operator $T : E \to F$ is limited p-convergent. □

Because of Theorem 4.1.14, we obtain the following variants of results in [46] for our setting of weak* p-convergent operators and the p-GPP:

Corollary 4.1.15 *Let E be a Banach lattice. Then the following assertions are equivalent:*

(1) Each weak p-convergent operator T from E into ℓ_∞ is limited p-convergent.*

(2) E has the p-GPP.

Corollary 4.1.16 *Let F be a σ-Dedekind complete Banach lattice. Then the following assertions are equivalent:*

(1) Each weak p-convergent operator T from ℓ_∞ into F is limited p-convergent.*

(2) F has order continuous norm.

4.2 Domination properties

The fact that every disjoint sequence in the solid hull of a relatively weakly compact subset of a Banach lattice converges weakly to zero, plays an important role in the proofs of many results concerning Dunford-Pettis operators and the Dunford-Pettis property on Banach lattices (and in more recent discussions of related versions of the same). We need a similar result in the context of this chapter, but currently we are only able to rely on the proof of Theorem 4.34 in [3] and some results in the literature concerning Banach lattices with non-trivial type to obtain the following lemma:

Lemma 4.2.1 *Let E be a Banach lattice with type q (with $1 < q \leq 2$) and let $p \geq q'$. Each disjoint sequence (x_n) in the solid hull of a relatively weakly compact subset W of E belongs to $\ell_p^{weak}(E)$.*

Proof Assume $p \geq q'$ (for $1 < q \leq 2$) and let a Banach lattice E with type q be given. Consider any disjoint sequence (x_n) in the solid hull of W. Pick a sequence $(y_n) \subset W$ such that $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$ and fix $0 \leq x^* \in E^*$. Using the disjointness of (x_n), and following the arguments of the proof of Theorem 4.34 in [3] (page 209), we can prove that there exists a sequence $(y_n^*) \in \ell_1^{weak}(E^*)$ such that $|x^*(x_n)| \leq y_n^*(y_n)$ for all n. Since E^* has cotype q' (cf. Proposition 11.10 in [30]), we may use Corollary 11.17 of [30] (page 224) to verify that

$$\ell_1^{weak}(E^*) \subseteq \ell_{q'}^{strong}(E^*) \subseteq \ell_p^{strong}(E^*).$$

For each $(\lambda_n) \in \ell_{p'}$ it follows that

$$\sum_{n=1}^{\infty} |\lambda_n x^*(x_n)| \leq \sum_{n=1}^{\infty} |\lambda_n||y_n^*(y_n)| \leq (\sup_n \|y_n\|) \sum_{n=1}^{\infty} |\lambda_n|\|y_n^*\| < \infty.$$

Since this holds for all $0 \leq x^* \in E^*$, it also follows that $(x^*(x_n)) \in \ell_p$ for all $x^* \in E^*$, i.e. that $(x_n) \in \ell_p^{weak}(E)$. □

Remark 4.2.2 *The reader is referred to [30] (Chapter 11) for the definitions and a discussion on the basics of the type and cotype properties on Banach spaces. In particular, in the context of our discussion of results in Banach lattices, the following remarks are of some importance:*

1. In Chapter 16 of [30] one finds a discussion of type and cotype in Banach lattices. Among others, the following results are proved:

(a) Let $1 < p \leq 2$ and let E be a Banach lattice with finite cotype. Then E has type p if and only if E^* has cotype p' (cf. Corollary 16.22 in [30]). Note that the dual space of any Banach space with type p has cotype p' (cf. Proposition 11.10 in [30]).

(b) Let $2 < q < \infty$. A Banach lattice E has cotype q if and only if id_E is $(q, 1)$-summing, thus, if and only if $\ell_1^{weak}(E) \subseteq \ell_q^{strong}(E)$. Therefore, if E is a Banach lattice with type q, where $1 < q \leq 2$, then for all $p \geq q'$ we have $\ell_1^{weak}(E^*) \subseteq \ell_p^{strong}(E^*)$.

2. It is well known (cf. for instance [2], Theorem 6.2.14, page 140) that:

(a) If $1 \leq p \leq 2$, then $L_p(\mu)$ has type p and cotype 2. The same is true for ℓ_p.

(b) If $2 < p < \infty$, then $L_p(\mu)$ has type 2 and cotype p. The same is true for ℓ_p.

3. A Banach space X is said to have the Orlicz property if the unconditionally summable sequences in X are strongly 2-summable. Since in this case X does not contain a copy of c_0, it is equivalent to saying that $\ell_1^{weak}(X) \subseteq \ell_2^{strong}(X)$. By a well-known result of Maurey, a Banach lattice has cotype 2 if and only if it has the Orlicz property.

We refer the reader to Chapter 2 for a discussion of Dunford-Pettis operators, in particular to the proof of the well-known Kalton-Saab Theorem which states that if a positive operator $S : E \to F$ between two Banach lattices (where F has order continuous norm) is dominated by a Dunford-Pettis operator, then S itself is Dunford-Pettis. Based on this proof and Lemma 4.2.1, we have the following domination result for p-convergent operators:

Theorem 4.2.3 *Let E, F be Banach lattices such that E has type $1 < q \leq 2$ and F has order continuous norm. If $T : E \to F$ is a positive p-convergent operator, where $p \geq q'$, then each positive operator $S : E \to F$ satisfying $0 \leq S \leq T$ is p-convergent itself.*

Proof Let $(x_n) \in \ell_p^{weak}(E)$ and let $\epsilon > 0$. Put $x = \sum_{n=1}^{\infty} 2^{-n}|x_n|$ and let E_x be the ideal generated by x in E. Clearly, the solid hull W of the (relatively weakly compact) set $\{x_n : n \in \mathbb{N}\}$ is contained in E_x. If (y_n) is any disjoint sequence in W, then by Lemma 4.2.1, $(y_n) \in \ell_p^{weak}(E)$. Thus, by the assumption on T, we conclude that $\|Ty_n\| \to 0$. This holds for all disjoint sequences in W. We now proceed as in the proof of Theorem 5.90 (in [3]): Using Theorem 4.36 in [3] (page 210) we conclude that there exists some $0 \le u \in E_x$ such that

$$\|T(|w| - u)^+\| < \epsilon, \ \forall w \in W.$$

In particular, we have

$$\|T(|x_n| - u)^+\| < \epsilon, \ \forall n \in \mathbb{N}.$$

Consider the operators $S, T : \overline{E_x} \to F$ (restrictions of S, T to the closure of E_x in E). Since x is a quasi-interior point of $\overline{E_x}$ (i.e $\overline{E_x}$ has quasi-interior points) we may use Theorem 4.87 in [3] to find operators $M_1, M_2, \ldots, M_k$ on $\overline{E_x}$ and positive operators $L_1, L_2, \ldots, L_k$ on F, satisfying

$$\| \, |S - \sum_{i=1}^{k} L_i T M_i| u\| \le \epsilon \quad \text{and} \quad 0 \le \sum_{i=1}^{k} L_i T M_i \le T \text{ on } \overline{E_x}.$$

Using the easy observation that a sequence lying in the closed subspace of a Banach space is weakly p-summable in the subspace if and only if it is weakly p-summable in the Banach space itself and the fact that each $M_i : \overline{E_x} \to \overline{E_x}$ is continuous (with respect to the induced norm of E), it follows that $(M_i(x_n))_n \in \ell_p^{weak}(E)$ for each $i = 1, \ldots, k$. Since T is p-convergent, it follows that

$$\lim_n \| \sum_{i=1}^{k} L_i T M_i(x_n)\| = 0.$$

Pick some m such that $\| \sum_{i=1}^{k} L_i T M_i(x_n)\| < \epsilon$ holds for all $n \ge m$. Now for $n \ge m$ we have

$$\begin{aligned}
\|Sx_n\| &\le \| \left(S - \sum_{i=1}^{k} L_i T M_i \right) x_n\| + \| \sum_{i=1}^{k} L_i T M_i(x_n)\| \\
&\le \| |S - \sum_{i=1}^{k} L_i T M_i| (|x_n| - u)^+\| + \| |S - \sum_{i=1}^{k} L_i T M_i| u\| + \varepsilon \\
&\le 2\|T(|x_n| - u)^+\| + \varepsilon + \varepsilon < 4\varepsilon,
\end{aligned}$$

and so $\|Sx_n\| \to 0$ as $n \to \infty$. □

Following the plan of Kalton and Saab in [48], where domination properties of weak Dunford-Pettis operators are studied, the authors in [22] prove that if E,F are Banach lattices such that F is σ-Dedekind complete, then if a positive operator $S : E \to F$ is dominated by a positive weak* Dunford-Pettis operator, then S itself is weak* Dunford-Pettis. Henceforth, we devote ourselves to a study of a similar domination property for the class of weak* p-convergent operators.

We shall need the following result:

Theorem 4.2.4 *(cf. [22], Theorem 2.2) Let E, F be Banach lattices such that E has type $1 < q \leq 2$ and F is σ-Dedekind complete. Let $T : E \to F$ be a positive weak* p-convergent operator. Then for every weakly p-summable sequence (x_n) in E^+ and every weak* null sequence (f_n) in F^*, we have*

$$|f_n|(Tx_n) \to 0 \text{ as } n \to \infty.$$

Proof Let $\varepsilon > 0$ be given. We claim that that there exist $0 \leq g \in F^*$ and $N \in \mathbb{N}$ such that

$$(\maltese) \qquad (|f_n| - g)^+(Tx_n) < \varepsilon_0$$

holds for all $n > N$. Suppose that ($\maltese$) is false. then there exists an $\varepsilon_0 > 0$ such that for each $0 \leq g \in F^*$ and each $N \in \mathbb{N}$, we have $(|f_k| - g)^+(Tx_n) \geq \varepsilon_0$ for at least one $k > N$. Put $g = 4|f_1|$ and $n_1 = 1$. Then there exists a natural number $n_2 > n_1$ satisfying

$$(|f_{n_2}| - 4|f_1|)^+(Tx_{n_2}) \geq \varepsilon_0.$$

Also, put $g = 4^2 \sum_{1=1}^{2} |f_{n_i}|$. Then

$$\left(|f_{n_3}| - 4^2 \sum_{i=1}^{2} |f_{n_i}|\right)^+ (Tx_{n_3}) \geq \varepsilon_0$$

for some natural number $n_3 > n_2$. Proceeding with an inductive argument, we can obtain a strictly increasing subsequence (n_k) of $\mathbb{N}$ such that

$$\left(|f_{n_{k+1}}| - 4^k \sum_{i=1}^{k} |f_{n_i}|\right)^+ (Tx_{n_{k+1}}) \geq \varepsilon_0$$

for all $k \in \mathbb{N}$. Let $f = \sum_{k=1}^{\infty} 2^{-k}|f_{n_k}|$ and put

$$g_{k+1} = \left(|f_{n_{k+1}}| - 4^k \sum_{i=1}^{k} |f_{n_i}|\right)^+, \quad \tilde{f}_{k+1} = \left(|f_{n_{k+1}}| - 4^k \sum_{i=1}^{k} |f_{n_i}| - 2^{-k} f\right)^+.$$

Note that $0 \leq g_{k+1} \leq \tilde{f}_{k+1} + 2^{-k} f$ and $g_{k+1}(Tx_{n_{k+1}}) \geq \varepsilon_0$ for any $k \in \mathbb{N}$. By Lemma 4.35 in [3], $(\tilde{f}_{k+1})$ is a disjoint sequence. Since $0 \leq \tilde{f}_{k+1} \leq |f_{n_{k+1}}|$ and $(f_{n_{k+1}}) \in c_0^{weak^*}(F^*)$, then by Lemma 2.1 in [21] we have $(\tilde{f}_{k+1}) \in c_0^{weak^*}(F^*)$. Since T is weak* p-convergent, it follows that $\tilde{f}_{k+1}(Tx_{n_{k+1}}) \to 0$. However,

$$\begin{aligned} 0 < \varepsilon_0 \leq g_{k+1}(Tx_{n_{k+1}}) &\leq (\tilde{f}_{k+1} + 2^{-k} f)(Tx_{n_{k+1}}) \\ &= \tilde{f}_{k+1}(Tx_{n_{k+1}}) + 2^{-k} f(Tx_{n_{k+1}}) \to 0. \end{aligned}$$

This leads to a contradiction and hence (✠) is true.

Now let $0 \leq g \in F^*$ and $N \in \mathbb{N}$ satisfy (✠). For all $n > N$, we have the inequalities

$$\begin{aligned} |f_n|(Tx_n) &= (|f_n| - g)^+ + (|f_n| \wedge g)(Tx_n) \\ &\leq (|f_n| - g)^+(Tx_n) + g(Tx_n) \\ &\leq \varepsilon + g(Tx_n). \end{aligned}$$

Since (x_n) is weakly p-summable in E^+, it follows that $\limsup |f_n|(Tx_n) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $|f_n|(Tx_n) \to 0$ as $n \to 0$. □

Notice that by the definition of weak* p-convergent operator, it follows that for the sequences (x_n) and (f_n) in the statement of Theorem 4.2.4 we already have $f_n(Tx_n) \to 0$ as $n \to \infty$. The important consequence of the σ-Dedekind completeness of F is that we have the stronger property $|f_n|(Tx_n) \to 0$ as $n \to \infty$. The role of the σ-Dedekind completeness of F is to assure (by a result in [21]) that both the sequences of positive parts and absolute values of a disjoint weak* null sequence in F^* are weak* null themselves.

Theorem 4.2.5 *Let $T : E \to F$ be a positive weak* p-convergent operator (for $1 \leq p < \infty$), where E, F are Banach lattices such that E is weak p-consistent and F is σ-Dedekind complete. If $0 \leq S \leq T$, then S is weak* p-convergent.*

Proof Let $(x_n) \in \ell_p^{weak}(E)$ and let $(f_n) \in c_0^{weak^*}(F^*)$. By assumption, $(|x_n|) \in \ell_p^{weak}(E)$ and $T : E \to F$ is weak* p-convergent. So, by Theorem 4.2.4, we have

$$|f_n|(T|x_n|) \to 0 \text{ as } n \to \infty.$$

This implies that $f_n^+(T|x_n|) \to 0$ and $f_n^-(T|x_n|) \to 0$ as $n \to \infty$. Therefore,

$$|f_n^+(Sx_n)| \leq f_n^+(|Sx_n|) \leq f_n^+(S|x_n|) \leq f_n^+(T|x_n|) \to 0.$$

Similarly, $|f_n^-(Sx_n)| \to 0$ as $n \to \infty$. This proves that $f_n(Sx_n) \to 0$ as $n \to \infty$. □

Corollary 4.2.6 *Let $T : E \to F$ be a positive weak* p-convergent operator (for $1 \leq p < \infty$), where E, F are Banach lattices such that E is an AM-space with unit and F is σ-Dedekind complete. If $0 \leq S \leq T$, then S is weak* p-convergent.*

To extend Theorem 4.2.5 so as to include more Banach lattices, we will make use of (adjusted) arguments from the paper [22], where similar domination results were proved for positive weak* Dunford-Pettis operators. The first result is a modification of Theorem 2.3 in [22]; the proof being largely a copy of the proof of the result in [22]. However, since in this case we have to make use of our results above, we nevertheless opt to discuss the proof here.

Theorem 4.2.7 *Let $T : E \to F$ be a positive weak* p-convergent operator, where E, F are Banach lattices such that E has type $1 < q \leq 2$ and F is σ-Dedekind complete. Given a weak p-summable sequence (z_n) in E (with $p \geq q'$), let W be the set of elements in the sequence (z_n). If $f_n \to 0$ weak* in F^*, then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ and some $w \in E^+$ lying in the ideal generated by W such that*

$$|f_n|(T(|x| - w)^+) < \epsilon,$$

for all $n > N$ and all $x \in W$.

Proof Suppose, to the contrary, that there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ and each $w \geq 0$ in the ideal generated by W, we can find a natural number $m > N$ and some $x_m \in W$ satisfying

$$|f_m|(T(|x_m| - w)^+) \geq \epsilon.$$

By an induction argument, we may then construct a sequence $(x_k) \subseteq W$ and an increasing sequence $n_1 < n_2 < n_3 < \dots$ of natural numbers such that

$$|f_{n_{k+1}}|(T(|x_{k+1}| - 4^k \sum_{i=1}^{k} |x_i|)^+) \geq \epsilon.$$

Putting $x = \sum_{k=1}^{\infty} 2^{-k}|x_k|$ and following the arguments in the proof of Theorem 2.2 in [22] we may thus construct a disjoint sequence (ν_{k+1}) in the solid hull of W, where

$$\nu_{k+1} = (|x_{k+1}| - 4^k \sum_{i=1}^{k} |x_i| - 2^{-k}x)^+.$$

By Lemma 4.2.1, $(\nu_{k+1}) \in \ell_p^{weak}(E)$. Hence, from Theorem 4.2.4 it follows that $|f_{n_{k+1}}|(Tv_{k+1}) \to 0$ if $k \to \infty$. Also, $|f_{n_{k+1}}|(Tx) \to 0$ if $k \to \infty$. However,

$$\begin{aligned} |f_{n_{k+1}}|(Tv_{k+1}) + 2^{-k}|f_{n_{k+1}}|(Tx) &= |f_{n_{k+1}}|(T(v_{k+1} + 2^{-k}x)) \\ &\geq |f_{n_{k+1}}|(T(|x_{k+1}| - 4^k \sum_{i=1}^{k} |x_i|)^+) \\ &\geq \epsilon > 0. \end{aligned}$$

Thus we have a contradiction! □

As the condition "E is weak p-consistent" in Theorem 4.2.5 is restrictive, we formulate the following version of the theorem:

Theorem 4.2.8 *Let E, F be Banach lattices such that E has type $1 < q \leq 2$ and F is σ-Dedekind complete. If $T : E \to F$ is a positive weak* p-convergent operator, where $p \geq q'$, then each positive operator $S : E \to F$ satisfying $0 \leq S \leq T$ is weak* p-convergent itself.*

Proof Let $(x_n) \in \ell_p^{weak}(E)$, $f_n \xrightarrow{w^*} 0$ in F^* and let $\epsilon > 0$ be given. Suppose the operators $S, T : E \to F$ satisfy the properties in the statement of the theorem. By the assumption on T, we have $f_n(Tx_n) \to 0$ and by Theorem 4.2.7, there exists $N_1 \in \mathbb{N}$ and some $u \in E^+$ lying in the ideal generated by (the relatively weakly compact) set $W := \{x_n : n \in \mathbb{N}\}$ such that

$$|f_n|(T(|x_n| - u)^+) < \epsilon, \ \forall n > N_1.$$

Since F is σ-Dedekind complete, there exists by Theorem 4.42 in [3] an element $0 \le g \in F^*$ lying in the ideal generated by (f_n) in F^* such that

$$(|f_n| - g)^+(Tu) < \epsilon$$

for all $n \in \mathbb{N}$.

Put $x = \sum_{n=1}^{\infty} 2^{-n}|x_n| \in E^+$ and let E_x be the ideal generated by x in E. Note that $u \in E_x$ and $W \subseteq E_x$. Restricting the positive operators S, T to E_x, we have two positive linear operators satisfying the conditions in Theorem 4.82 in [3] on the AM-space E_x with unit x. Therefore there exist positive multiplication operators $M_1, M_2, \ldots, M_k$ on $\overline{E_x}$ (extensions of positive multiplication operators on E_x to its closure in E) and order projections $P_1, P_2, \ldots, P_k$ on F'', satisfying

$$\left\langle g, \left| S - \sum_{i=1}^{k} P_i T M_i \right| u \right\rangle < \epsilon \text{ and } 0 \le \sum_{i=1}^{k} P_i T M_i \le T$$

on $\overline{E_x}$. Using that a sequence is weakly p-summable in $\overline{E_x}$ if and only if it is weakly p-summable in E and the fact that each $M_i : \overline{E_x} \to \overline{E_x}$ is continuous (with respect to the induced norm of E), it follows that $(M_i(x_n))_n \in \ell_p^{weak}(E)$ for each $i = 1, \ldots, k$.

Put $R = \left| S - \sum_{i=1}^{k} P_i T M_i \right|$, then

$$\langle g, Ru \rangle < \varepsilon \quad \text{and } R = \left| S - \sum_{i=1}^{k} P_i T M_i \right| \le S + \sum_{i=1}^{k} P_i T M_i \le 2T.$$

For each $n > N_1$,

$$\begin{aligned}
\langle |f_n|, R|x_n| \rangle &= \langle |f_n|, R(|x_n| - u)^+ \rangle + \langle |f_n|, R(|x_n| \wedge u) \rangle \\
&\le \langle |f_n|, R(|x_n| - u)^+ \rangle + \langle |f_n|, Ru \rangle \\
&\le 2|f_n|T((|x_n| - u)^+) + \langle |f_n|, Ru \rangle \\
&\le 2|f_n|T((|x_n| - u)^+) + \langle (|f_n| - g)^+, Ru \rangle + \langle g, Ru \rangle \\
&\le 2|f_n|T((|x_n| - u)^+) + 2(|f_n| - g)^+ T(u) + \langle g, Ru \rangle \\
&< 2\varepsilon + 2\varepsilon + \varepsilon \\
&= 5\varepsilon.
\end{aligned}$$

This implies that

$$\begin{aligned} |f_n(Sx_n)| &\leq |\langle f_n, (S - \sum_{i=1}^{k} P_i T M_i) x_n \rangle| + \sum_{i=1}^{k} |\langle f_n, P_i T M_i x_n \rangle| \\ &\leq |\langle |f_n|, |S - \sum_{i=1}^{k} P_i T M_i| |x_n| \rangle| + \sum_{i=1}^{k} |\langle f_n, P_i T M_i x_n \rangle| \\ &= \langle |f_n|, R|x_n| \rangle + \sum_{i=1}^{k} |\langle f_n, P_i T M_i x_n \rangle| \\ &< 5\varepsilon + \sum_{i=1}^{k} |\langle f_n, P_i T M_i x_n \rangle| \end{aligned}$$

holds for all $n > N_1$.

Note that each P_i is an order projection on F^{**}. For each $f \in F^*$, we define $Q_i f$ by

$$(Q_i f)y = \langle f, P_i y \rangle$$

for all $y \in F$. Observe that $Q_i f \in F^*$ and $Q_i : F^* \to F^*$ is a bounded linear operator.
Assume that $(f_n) \in c_0^{weak^*}(F^*)$ and let A_F denote the ideal generated by F in F^{**}. Then $f_n \overset{\sigma(F^*, A_F)}{\longrightarrow} 0$ in F, since F is σ-Dedekind complete (cf. [3], Theorem 4.43). Recall that P_i is an order projection on F^{**}. Given $y \in F^+$, since $0 \leq P_i y \leq y$, we have that $P_i y \in A_F$. Hence,

$$\langle y, Q_i f_n \rangle = \langle f_n, P_i y \rangle \xrightarrow[\infty]{n} 0,$$

which implies the sequential weak*-continuity of Q_i.

Since $(M_i(x_n))_n \in \ell_p^{weak}(E)$ for each $i = 1, \ldots, k$ and T is weak* p-convergent, it therefore follows that

$$\sum_{i=1}^{k} |\langle f_n, P_i T M_i x_n \rangle| = \sum_{i=1}^{k} |\langle T M_i x_n, Q_i f_n \rangle| \xrightarrow{n} 0.$$

Thus, $f_n(Sx_n) \to 0$ if $n \to \infty$. Since this holds for arbitrary $(x_n) \in \ell_p^{weak}(E)$ and $(f_n) \in c_0^{weak^*}(F^*)$, we see that S has to be weak* p-convergent. □

Chapter 5

On sequentially limited operators

For a Banach space Y we let

$$\ell_p^{strong}(Y) = \{(y_n) \in Y^{\mathbb{N}} : (\|y_n\|) \in \ell_p\}, \text{ with norm } \|(y_n)\|_p = \left(\sum_{i=1}^{\infty} \|y_i\|^p\right)^{1/p}.$$

In the paper [48] the authors introduce the operator p-summable sequences in a Banach space X and among some applications with respect to p-limited sets and the p-Dunford-Pettis-property, they introduce the sequentially p-limited operators which map weakly p-summable sequences to operator p-summable sequences. In a brief discussion of sequentially p-limited operators, they introduce a norm ℓt_p on each vector space $Lt_p(X, Y)$ of sequentially p-limited operators as X,Y run through the family of all Banach spaces and show that $(Lt_p, \ell t_p)$ is a normed operator ideal. They could not verify the completeness of this normed ideal and therefore followed a completion procedure to obtain a Banach operator ideal.

Inspired by the paper [48], especially by the Banach ideal property of $(Lt_p, \ell t_p)$, we introduce the general concept of "operator $[Y, p]$-summable sequence" in a Banach space X, consider the vector space $Y_p(X)$ of all operator $[Y, p]$-summable sequences in X and introduce a norm on this space. We then prove that $Y_p(X)$ is a Banach space. The results of the general setting are then applied to the special setting of operator p-summable sequences in a Banach space X. We then consider the sequentially p-limited operators and following standard techniques for p-summing operators, we prove that given

any pair X, Y of Banach spaces, then the normed space $(Lt_p(X,Y), \ell t_p(\cdot))$ is a Banach space. Thus, the pair $(Lt_p, \ell t_p)$ is a complete normed operator ideal. We then study the normed operator ideal $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ of operators $T : X \to Y$ so that for the scalar Banach sequence space Λ we have $ST \in \mathcal{A}(X, \Lambda)$ for all $S \in \mathcal{L}(Y, \Lambda)$, where X,Y run through the family of all Banach spaces and where $(\mathcal{A}, \alpha)$ is a given normed operator ideal. The corresponding normed operator ideal $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ is studied and it is shown that if $(\mathcal{A}, \alpha)$ is a Banach operator ideal, then so is $(\mathcal{A}_\Lambda, \alpha_\Lambda)$. These results may also be obtained by using the operator ideal approach and we also consider some other classes of sequentially limited operators, which are special cases of the operator ideals, and prove some multiplication (or composition) results.
Please note that the main results in this chapter was published the previous year (cf. [38]).

5.1 Operator p-summable sequences

Definition 5.1.1 *Let X, Y be given Banach spaces and let $1 \le p < \infty$. A sequence (x_n) in X is called operator $[Y,p]$-summable if $\sum\limits_{n=1}^{\infty} \|Tx_n\|^p < \infty$ for all $T \in \mathcal{L}(X,Y)$, i.e. if $(Tx_n) \in \ell_p^{strong}(Y)$ for all $T \in \mathcal{L}(X,Y)$.*

We can extend the Definition 5.1.1 above to include the case when $p = \infty$ by adopting the convention that "operator $[Y, \infty]$-summable sequence (x_n) in X" will mean that for each $T \in \mathcal{L}(X,Y)$ we have $\|Tx_n\| \xrightarrow[\infty]{n} 0$ (i.e. $(Tx_n) \in c_0^{strong}(Y)$, $\forall T \in \mathcal{L}(X,Y)$).

Let

$$Y_p(X) := \{(x_i) \in X^{\mathbb{N}} : (x_i) \text{ is operator } [Y,p]\text{-summable}\}.$$

For a given $(x_i) \in Y_p(X)$, we define an operator

$$\Theta : \mathcal{L}(X,Y) \to \ell_p^{strong}(Y) : T \mapsto (Tx_n).$$

We show that Θ has closed graph:

Let $T_n \xrightarrow[\infty]{n} T$ in $\mathcal{L}(X,Y)$ and $\Theta(T_n) = (T_n x_j)_j \xrightarrow[\infty]{n} (y_j) \in \ell_p^{strong}(Y)$, i.e.

$$\left(\sum_{j=1}^{\infty} \|T_n x_j - y_j\|^p\right)^{\frac{1}{p}} \xrightarrow[\infty]{n} 0.$$

In particular we have

$$\|T_n x_j - y_j\|_Y \xrightarrow[\infty]{n} 0$$

for each $j = 1, 2, \ldots$

However, we also have

$$\|T_n x_j - T x_j\|_Y \xrightarrow[\infty]{n} 0$$

for each $j = 1, 2, \ldots$

Thus, $Tx_j = y_j$ for $j = 1, 2, \ldots$

Therefore, $(y_j) = (Tx_j)$ and so $\Theta(T_n) \xrightarrow[\infty]{n} (Tx_j) = \Theta(T)$.

The underlying spaces being Banach spaces, it thus follows that Θ is a bounded linear operator. Therefore,

$$\sup\left\{\left(\sum_{n=1}^{\infty} \|Tx_n\|^p\right)^{1/p} : T \in \mathcal{L}(X,Y), \|T\| \leq 1\right\} = \|\Theta\| < \infty.$$

Since this is true for each $(x_i) \in Y_p(X)$, we may define $\|\cdot\|_{Y_p} : Y_p(X) \to \mathbb{R}$ by

$$\|(x_i)\|_{Y_p} := \sup\left\{\left(\sum_{n=1}^{\infty} \|Tx_n\|^p\right)^{1/p} : T \in \mathcal{L}(X,Y), \|T\| \leq 1\right\}.$$

It is easy to verify that $\|\cdot\|_{Y_p}$ defines a norm on the vector space $Y_p(X)$. For instance, if $\|(x_i)\|_{Y_p} = 0$, then given arbitrary $x^* \in U_{X^*}$ and $y \in U_Y$, we may consider the operator

$$x^* \otimes y : X \to Y : x \mapsto \langle x, x^* \rangle y$$

(with norm ≤ 1) and realise from the definition of $\|\cdot\|_{Y_p}$, that

$$|\langle x_n, x^* \rangle| \leq \|(x^* \otimes y)x_n\| \leq \left(\sum_{i=1}^{\infty} \|(x^* \otimes y)x_i\|^p\right)^{1/p} \leq \|(x_i)\|_{Y_p} = 0.$$

This implies $|\langle x_n, x^* \rangle| = 0$ for all $x^* \in X^*$ and all $n \in \mathbb{N}$. Thus (x_i) is a null sequence.

Lemma 5.1.2 *Suppose* $x_n \xrightarrow[\infty]{n} x$ *in* $(Y_p(X), \|\cdot\|_{Y_p})$, *where* $x_n = (x_{n,j})_j$ *and* $x = (x_j)$. *Then for each* $j \in \mathbb{N}$, *we have* $x_{n,j} \xrightarrow[\infty]{n} x_j$ *in* X.

Proof Let $y \in Y$ with $\|y\| = 1$. Then,

$$\begin{aligned} \|x_{n,j} - x_j\| &= \sup_{\|x^*\|\leq 1} |\langle x_{n,j} - x_j, x^* \rangle| \\ &\leq \sup_{\|x^*\|\leq 1} \|(\|(x^* \otimes y)(x_{n,j} - x_j)\|)_j\|_{\ell_p^{strong}(Y)} \\ &\leq \|(x_{n,j})_j - (x_j)\|_{Y_p} \xrightarrow[\infty]{n} 0 \end{aligned}$$

for each $j \in \mathbb{N}$. □

Using Lemma 5.1.2 and the completeness of the space X, it is easy to verify that $(Y_p(X), \|\cdot\|_{Y_p})$ is a complete normed space:

Theorem 5.1.3 $(Y_p(X), \|\cdot\|_{Y_p})$ *is a Banach space.*

Proof Given a Cauchy sequence (x_n) in $Y_p(X)$, and arbitrary $\epsilon > 0$, let $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\|_{Y_p} < \epsilon$ for all $m, n \geq n_0$. If $x_n = (x_{n,j})_j$, it follows as in the proof of Lemma 5.1.2 that

$$\|x_{n,j} - x_{m,j}\| \leq \|x_n - x_m\|_{Y_p} \xrightarrow[\infty]{m,n} 0,$$

showing that we may put $x_j := \lim_n x_{n,j} \in X$. Letting $x = (x_j)$, it follows by a standard argument that $x \in Y_p(X)$ and $\|x_n - x\|_{Y_p} \leq \epsilon$ for all $n \geq n_0$. □

In the paper [48] (and elsewhere in the literature), the well-known concept of "limited set" in a Banach space is generalized to introduce the so-called "p-limited" sets. A subset D of a Banach space X is said to be *p-limited* ($1 \leq p < \infty$) if for each weak* p-summable sequence (x_n^*) in X^* there exists a sequence $(\lambda_i) \in \ell_p$ such $|\langle x, x_n^* \rangle| \leq \lambda_n$ for each $n \in \mathbb{N}$ and all $x \in D$, i.e if and only if for each weak* p-summable sequence (x_n^*) in X^*, we have $(\sup_{x\in D} |\langle x, x_n^* \rangle|)_n \in \ell_p$.

Replacing Y in Definition 5.1.1 by ℓ_p, we agree to use the phrase "operator p-summable" instead of "operator $[\ell_p, p]$-summable", thereby recalling a definition from the paper [48]:

Definition 5.1.4 *(cf. [48]) Let* $1 \leq p < \infty$. *A sequence* (x_n) *is called operator* p*-summable if* $(Tx_n) \in \ell_p^{strong}(\ell_p)$ *for all* $T \in \mathcal{L}(X, \ell_p)$.

It should be clear that a sequence (x_n) in X is operator p-summable if and only if $((\langle x_n, x_i^*\rangle)_i)_n \in \ell_p^{strong}(\ell_p)$ for all $(x_i^*) \in \ell_p^{weak^*}(X^*)$. For an operator p-summable (x_n) in X and $x^* \in X^*$ it therefore follows that $((\langle x_n, x^*\rangle, 0, 0, \dots))_n \in \ell_p^{strong}(\ell_p)$, i.e. $\sum_n |\langle x_n, x^*\rangle|^p < \infty$, showing that $(x_n) \in \ell_p^{weak}(X)$. Therefore, we may consider the operator $E_{(x_i)} : \ell_{p'} \to X$ as before. By Proposition 2.4 in [48] a sequence (x_n) in a Banach space X is operator p-summable if and only if $(x_n) \in \ell_p^{weak}(X)$ and $E_{(x_i)}(U_{\ell_{p'}})$ is a p-limited set.

Let $\ell_p^o(X)$ denote the vector space of all operator p-summable sequences in the Banach space X (i.e. following the notation of Definition 5.1.1, we put $\ell_p^o(X) = (\ell_p)_p(X)$). If for $(x_i) \in \ell_p^o(X)$ we let

$$\|(x_i)\|_p^o := \sup\left\{\left(\sum_n \|Tx_n\|_p^p\right)^{1/p} : T \in \mathcal{L}(X, \ell_p), \|T\| \le 1\right\},$$

then it follows from Theorem 5.1.3 that:

Theorem 5.1.5 *$(\ell_p^o(X), \|\cdot\|_p^o)$ is a Banach space.*

Theorem 5.1.6 *Let $1 \le p < \infty$. We have the following continuous (norm ≤ 1) inclusions:*

$$\ell_p^{strong}(X) \subseteq \ell_p^o(X) \subseteq \ell_p^{weak}(X).$$

Proof For $(x_i) \in \ell_p^{strong}(X)$, it is clear that $(Tx_i) \in \ell_p^{strong}(\ell_p)$ for all $T \in \mathcal{L}(X, \ell_p)$ and that

$$\|(x_i)\|_p^o = \sup_{T \in U_{\mathcal{L}(X,\ell_p)}} \left(\sum_{i=1}^\infty \|Tx_i\|^p\right)^{1/p} \le \left(\sum_{i=1}^\infty \|x_i\|^p\right)^{1/p} = \|(x_i)\|_p^{strong}.$$

Also, for $(x_i) \in \ell_p^o(X)$ and $x^* \in X^*$, it is clear that $|\langle x_n, x^*\rangle| = \|(x^* \otimes e_1)x_n\|_p$ and so we have

$$\sup_{x^* \in U_{X^*}} \left(\sum_{n=1}^\infty |\langle x_n, x^*\rangle|^p\right)^{1/p} = \sup_{x^* \in U_{X^*}} \left(\sum_{n=1}^\infty \|(x^* \otimes e_1)x_n\|_p^p\right)^{1/p}.$$

Using that the bounded linear operator $x^* \otimes e_1 : X \to \ell_p$ has norm ≤ 1, it thus follows that $\|(x_i)\|_p^{weak} \le \|(x_i)\|_p^o$. □

Remark 5.1.7 *Let $1 \leq p < \infty$. Recall that an operator $T \in \mathcal{L}(X,Y)$ is said to be p-summing if $(Tx_n) \in \ell_p^{strong}(Y)$ for all $(x_n) \in \ell_p^{weak}(X)$. The vector space $\Pi_p(X,Y)$ of all p-summing operators is a Banach space with respect to the norm*

$$\pi_p(T) := \sup\{\|(Tx_n)\|_p : \|(x_n)\|_p^{weak} \leq 1\}.$$

For a detailed discussion of p-summing operators, the reader is referred to the book [30]. In [48] the authors introduce and study the so called weak p-spaces. A Banach space X is called a weak p-space (or X is said to have the p-Dunford-Pettis property) if $\ell_p^o(X) = \ell_p^{weak}(X)$. This is the case if and only if $\Pi_p(X,\ell_p) = \mathcal{L}(X,\ell_p)$ (cf. [48], Proposition 3.1). It is therefore immediately clear that ℓ_p itself is not a weak p-space. Moreover, it is shown in [48] that ℓ_p (for $1 < p < \infty$) is in fact not a weak r-space for any $r > 1$.

By Theorem 8.3.1 in [2] (page 213) every $T \in \mathcal{L}(L_1(\mu),\ell_2)$ is absolutely summing and therefore also 2-summing. Thus the space $L_1(\mu)$ is a weak 2-space. Since ℓ_1 is an $L_1(\mu)$-space for a suitable measure μ, the same theorem in [2] also holds for operators $T : \ell_1 \to \ell_2$. Thus, in contrast with the spaces ℓ_p for $1 < p < \infty$, the space ℓ_1 is a weak 2-space. If X is an infinite dimensional reflexive Banach space, then all p-summing operators on X are compact (cf.[2], Corollary 8.2.15, page 211). Therefore $K(X,\ell_p) = \mathcal{L}(X,\ell_p)$ if X is a reflexive weak p-space. The equality $K(X,\ell_p) = \mathcal{L}(X,\ell_p)$ holds if and only if $\ell_{p,c}^{weak^*}(X^*) = \ell_p^{weak^*}(X^*)$ and this is the case if and only if $\ell_p^{weak^*}(X^*) \subset c_0^{strong}(X^*)$ (cf. for instance [12] and [35] for these facts).

5.2 Sequentially p-limited operators

In [48] (Definition 4.1) an operator $T \in \mathcal{L}(X,Y)$ is said to be sequentially p-limited if it maps weakly p-summable sequences to operator p-summable sequences, i.e.

Definition 5.2.1 *(cf. [48], Definition 4.1) Let $1 \leq p < \infty$. An operator $T \in \mathcal{L}(X,Y)$ is called sequentially p-limited if $(Tx_n) \in \ell_p^o(Y)$ for all $(x_n) \in \ell_p^{weak}(X)$.*

It is clear from the definition and Remark 5.1.7 that id_X is sequentially p-limited if and only if X is a weak p-space. An operator $T : X \to Y$ is sequentially p-limited if and only if RT is p-summing for all $R \in \mathcal{L}(Y,\ell_p)$.

Refer to [48] (Theorem 4.4, p.435) for this fact. Following [48] we let

$$Lt_p(X,Y) := \{T \in \mathcal{L}(X,Y) : T \text{ is sequentially } p\text{-limited}\}.$$

The authors (in [48]) define a norm on $Lt_p(X,Y)$ by

$$\ell t_p(T) := \sup\{\pi_p(RT) : R \in \mathcal{L}(Y,\ell_p) \text{ and } \|R\| \leq 1\}$$

and mention that it is routine to show that the pair $(Lt_p, \ell t_p)$ so defined is a normed operator ideal. However, the authors also note in [48] that they could not show that $(Lt_p, \ell t_p)$ is a Banach operator ideal. In order to obtain a Banach operator ideal, they settled for taking the completions of isometric copies of the components $Lt_p(X,Y)$ of the ideal $(Lt_p, \ell t_p)$ in the corresponding Banach spaces $\mathcal{L}(\mathcal{L}(Y,\ell_p), \Pi_p(X,\ell_p))$.

Here is what happens in [48]: For each $T \in \mathcal{L}(X,Y)$ the authors consider the operator $\varphi_T : \mathcal{L}(Y,\ell_p) \to \mathcal{L}(X,\ell_p)$, given by $\varphi_T(S) = ST$, and note that $T \mapsto \varphi_T$ is a linear isometry from $\mathcal{L}(X,Y)$ into $\mathcal{L}(\mathcal{L}(Y,\ell_p), \mathcal{L}(X,\ell_p))$ for $1 \leq p \leq \infty$ (where in the case of $p = \infty$, the space ℓ_∞ is replaced by c_0). Then $T \in Lt_p(X,Y)$ if and only if $\varphi_T(\mathcal{L}(Y,\ell_p)) \subset \Pi_p(X,\ell_p)$ and $\ell t_p(T) = \|\varphi_T\|$; here $\|\varphi_T\|$ denotes the operator norm of φ_T considered as an element of $\mathcal{L}(\mathcal{L}(Y,\ell_p), \Pi_p(X,\ell_p))$. This can be done, since the Closed Graph Theorem implies that if $T \in Lt_p(X,Y)$, then

$$\sup_{R \in U_{\mathcal{L}(Y,\ell_p)}} \pi_p(RT) < \infty.$$

A discussion of this fact in the general setting of operator ideals will follow later in the chapter. Finally, the authors consider the completion (closure) of the set $\{\varphi_T : T \in Lt_p(X,Y)\}$ in the complete space $\mathcal{L}(\mathcal{L}(Y,\ell_p), \Pi_p(X,\ell_p))$ and denote this completion (also) by $Lt_p(X,Y)$. In this way a Banach operator ideal is obtained (cf. [48], Proposition 4.7).

The presence of the Banach operator ideal (Π_p, π_p) in the definition of $(Lt_p, \ell t_p)$ suggests a different approach in the study of sequentially p-limited operators via the theory of p-summing operators. Based on our discussion of the sequence space $\ell_p^o(X)$, we are now ready to discuss the completeness of the normed space $(Lt_p(X,Y), \ell t_p(\cdot))$ in the following theorem.

Theorem 5.2.2 *Let $1 \leq p < \infty$ and let X, Y be Banach spaces. The space $(Lt_p(X,Y), \ell t_p(\cdot))$ of sequentially p-limited operators is a Banach space. Thus, $(Lt_p, \ell t_p)$ is a Banach operator ideal.*

Proof We associate with each $T \in Lt_p(X,Y)$ the operator

$$\widehat{T} : \ell_p^{weak}(X) \to \ell_p^o(Y) : (x_i) \mapsto (Tx_i)$$

and show that $\widehat{T}$ has closed graph: Let $(x_{n,i})_i \xrightarrow[\infty]{n} (x_i)$ in $\ell_p^{weak}(X)$ and suppose that $\widehat{T}((x_{n,i})_i) = (Tx_{n,i})_i \xrightarrow[\infty]{n} (y_i)$ in $\ell_p^o(Y)$. Then we have:

(i) For $\epsilon > 0$ given, there exists $n_0 \in \mathbb{N}$ such that

$$\sup\left\{\left(\sum_{i=1}^{\infty}\|R(Tx_{n,i}-y_i)\|_p^p\right)^{1/p} : R \in U_{\mathcal{L}(Y,\ell_p)}\right\} < \epsilon, \quad \forall n \geq n_0$$

$$\Longrightarrow \sup_{\|y^*\|\leq 1}\left(\sum_{i=1}^{\infty}\|(y^*\otimes e_1)(Tx_{n,i}-y_i)\|_p^p\right)^{1/p} < \epsilon, \quad \forall n \geq n_0$$

$$\Longrightarrow \|Tx_{n,i}-y_i\| < \epsilon, \quad \forall n \geq n_0 \text{ and } \forall i \in \mathbb{N}.$$

(ii)

$$\sup_{\|x^*\|\leq 1}\left(\sum_{i=1}^{\infty}|\langle x_{n,i}-x_i, x^*\rangle|^p\right)^{1/p} \xrightarrow[\infty]{n} 0.$$

Thus $x_{n,i} \xrightarrow[\infty]{n} x_i$ weakly in X for each $i \in \mathbb{N}$. Since T is weakly continuous, it follows that $Tx_{n,i} \xrightarrow[\infty]{n} Tx_i$ weakly in Y for each $i \in \mathbb{N}$.

From (i) and (ii) above we have $y_i = Tx_i$ for $i = 1, 2, \ldots$, i.e.

$$\widehat{T}((x_{n,i})_i) \to \widehat{T}((x_i)) \text{ as } n \to \infty.$$

This shows that $\widehat{T}$ has closed graph, and hence is bounded. Note that

$$\|\widehat{T}\| = \sup_{\|(x_i)\|_p^{weak}\leq 1}\|(Tx_i)\|_p^o.$$

Also,

$$\sup_{R\in U_{\mathcal{L}(Y,\ell_p)}}\pi_p(RT) = \sup_{\|(x_i)\|_p^{weak}\leq 1}\sup\{\|(RTx_i)\|_p : R \in U_{\mathcal{L}(Y,\ell_p)}\} = \|\widehat{T}\|,$$

showing that $\ell t_p(T) = \|\widehat{T}\|$. From this discussion it is clear that the mapping

$$\Phi : Lt_p(X,Y) \to \mathcal{L}(\ell_p^{weak}(X), \ell_p^o(Y)) : T \mapsto \widehat{T}$$

is an isometry which associates the space $(Lt_p(X,Y), \ell t_p(\cdot))$ isometrically with a subspace of the Banach space $\mathcal{L}(\ell_p^{weak}(X), \ell_p^o(Y))$. We denote the range space $\Phi(Lt_p(X,Y))$ by $\widehat{L}t_p(X,Y)$ and prove that it is a closed subspace of the complete normed space $\mathcal{L}(\ell_p^{weak}(X), \ell_p^o(Y))$: Consider any sequence $(\widehat{T}_n)$ in $\widehat{L}t_p(X,Y)$ which converges in operator norm to some operator $S \in \mathcal{L}(\ell_p^{weak}(X), \ell_p^o(Y))$. If $(x_i) \in \ell_p^{weak}(X)$ and $S((x_i)) = (y_i) \in \ell_p^o(Y)$, then

$$\|(T_n x_i) - (y_i)\|_p^o = \|\widehat{T}_n((x_i)) - (y_i)\|_p^o \xrightarrow[\infty]{n} 0,$$

where $T_n \in Lt_p(X,Y)$ such that $\Phi(T_n) = \widehat{T}_n$ for all $n \in \mathbb{N}$. By Lemma 5.1.2, it thus follows that $T_n x_j \xrightarrow[\infty]{n} y_j$ for all $j \in \mathbb{N}$. This implies that

$$S((x, 0, 0, \cdots)) = (\lim_n T_n x, 0, 0, \cdots), \quad \forall x \in X.$$

Therefore, if we put $Tx = \lim_n T_n x$ for each $x \in X$, then T is a bounded linear operator and given any $(x_i) \in \ell_p^{weak}(X)$ and $S((x_i)) = (y_i)$, we have $Tx_i = y_i$ for all $i \in \mathbb{N}$. Thus $T \in Lt_p(X,Y)$ and $S = \widehat{T} \in \widehat{L}t_p(X,Y)$, showing that $\widehat{L}t_p(X,Y)$ is a closed subspace of the Banach space $\mathcal{L}(\ell_p^{weak}(X), \ell_p^o(Y))$, i.e. $Lt_p(X,Y)$ is a Banach space for all Banach spaces X, Y. □

Let $1 < p < \infty$. In this case, using Proposition 2.19 in [30] (page 50), it is easy to see that if the second dual operator $T^{**} : X^{**} \to Y^{**}$ of the operator $T \in \mathcal{L}(X,Y)$ is sequentially p-limited, then so is T. Moreover, we also have:

Proposition 5.2.3 *Let $1 < p < \infty$. If an operator $T : X \to Y$ is sequentially p-limited and weakly compact, then so is its second dual T^{**}.*

Proof Assume that $T : X \to Y$ is sequentially p-limited and weakly compact. Since T is weakly compact, it follows from Theorem 5.5 in [24] (page 185) that T^{**} is weakly compact and that $T^{**}(X^{**}) \subseteq Y$. Let $S \in \mathcal{L}(Y^{**}, \ell_p)$ and denote the canonical embedding (evaluation) from Y into Y^{**} by C_Y. Recall that C_Y^* defines a canonical projection from Y^{***} to Y^*. Using the above information we get

$$\langle ST^{**}x^{**}, \gamma\rangle = \langle S^{**}C_Y^{**}T^{**}x^{**}, \gamma\rangle$$

for all $x^{**} \in X^{**}, \gamma \in \ell_{p'}$ and thus that

$$ST^{**} = S^{**}C_Y^{**}T^{**} = (SC_YT)^{**}.$$

Since SC_YT is p-summing, it follows from Proposition 2.19 in [30] that ST^{**} is p-summing. This completes the proof. □

There are sequentially p-limited operators which are not weakly compact. Refer to the discussion in Remark 5.1.7 above. Each bounded linear operator from ℓ_1 to ℓ_2 is absolutely summing, hence by the Inclusion Theorem (cf. [30], Theorem 2.8, page 39) each bounded linear operator from ℓ_1 to ℓ_2 is p-summing for all $1 \leq p < \infty$, in particular, each $S \in \mathcal{L}(\ell_1, \ell_2)$ is 2-summing. Therefore, the identity $id_{\ell_1} : \ell_1 \to \ell_1$ is sequentially 2-limited, but not weakly compact. This argument, of course, will also imply that for all nonreflexive Banach spaces X such that $\mathcal{L}(X, \ell_p) = \Pi_p(X, \ell_p)$ (i.e. for all nonreflexive weak-p spaces) the identity id_X on X is sequentially p-limited but not weakly compact. Both ℓ_1 and c_0 are examples of nonreflexive weak 2-spaces (cf. [48]). By an application of Grothendieck's Inequality, it follows that for each compact Hausdorff space K and any measure μ, we have $\mathcal{L}(C(K), L_p(\mu)) = \Pi_2(C(K), L_p(\mu))$ if $1 \leq p \leq 2$. In particular, this yields $\mathcal{L}(C(K), \ell_2) = \Pi_2(C(K), \ell_2)$, i.e. that $C(K)$ is a nonreflexive weak 2-space (cf. [30], Theorem 3.5 for the details). This is also true for all Banach spaces X such that X^{**} is a $C(K)$ space.

The immediate question arising from Proposition 5.2.3 is when a sequentially p-limited operator $T : X \to Y$ will be weakly compact. In the following lemma we list some conditions (by no means all possible conditions) from the literature which imply that each bounded linear operator is weakly compact.

Lemma 5.2.4 *For two Banach spaces X and Y, each $T \in \mathcal{L}(X, Y)$ is weakly compact if:*

(i) Either X or Y is reflexive.

(ii) $X = C(K)$ for some compact Hausdorff space K and no closed subspace of Y is isomorphic to c_0 (cf. [2], Corollary 5.5.4, page 120).

(iii) X does not contain a copy of ℓ_1 and $Y = L_1$, where L_1 denotes the space $L_1([0,1], \lambda)$ and λ is Lebesgue measure on $[0,1]$ (cf. [2], page 125).

(iv) *If X has type $r > 1$ and $Y = L_1(\mu)$ (for some σ-finite measure μ), for in this case each T factors through the reflexive space $L_q(\mu)$ for all $1 < q < r$ (cf. [2], Theorem 7.1.8, page 172).*

(v) *X has type 2 and Y has cotype 2, for in this case each T factors through a Hilbert space by the well-known Kwapień-Maurey Theorem (cf. [2], Theorem 7.4.2, page 187).*

(vi) *X^* has cotype 2 and $Y = L_1$, for in this case each T factors through a Hilbert space (cf. [2], Theorem 8.1.7, page 203).*

(vii) *X^* has cotype 2, Y has cotype 2 and either X or Y has the approximation property, for in this case each T factors through a Hilbert space by Pisier's Abstract Grothendieck Theorem (cf. [2], Theorem 8.1.8, page 204).*

From Proposition 5.2.3 and Lemma 5.2.4 we conclude that:

Corollary 5.2.5 *Let $1 < p < \infty$. If the Banach spaces X and Y satisfy any one of the conditions (i) to (vii) in Lemma 5.2.4, then if $T : X \to Y$ is sequentially p-limited, so is T^{**}.*

5.3 An operator ideal approach

The reader is referred to [54] for information on operator ideals. Consider a Banach operator ideal $(\mathcal{A}, \alpha)$. Fix a Banach sequence space $(\Lambda, \| \cdot \|_\Lambda)$ which contains the set ϕ of all sequences having only a finite number of non-zero terms and for which $\|e_n\|_\Lambda = 1$ for all $n \in \mathbb{N}$. Clearly, $\Lambda = \ell_p$ (for $1 \leq p \leq \infty$) and $\Lambda = c_0$ satisfy these properties. However, there are more Banach sequence spaces with these properties (see for instance Remark 5.3.11 at the end of this section). With the vector space $\mathcal{A}(X, Y)$ we associate

$$\mathcal{A}_\Lambda(X, Y) := \{T \in \mathcal{L}(X, Y) : ST \in \mathcal{A}(X, \Lambda), \forall S \in \mathcal{L}(Y, \Lambda)\}.$$

From the operator ideal properties of $\mathcal{A}$ it is easily verified that $\mathcal{A}_\Lambda$ also defines an operator ideal.

For $T \in \mathcal{L}(X, Y)$ we let

$$\phi_T : \mathcal{L}(Y, \Lambda) \to \mathcal{L}(X, \Lambda) : S \mapsto ST.$$

Then ϕ_T is a bounded linear operator for which $\|\phi_T\| \leq \|T\|$ is clear from its definition. On the other hand,

$$\|\phi_T\| \geq \sup_{\|y^*\|\leq 1} \|(y^* \otimes e_1) \circ T\| = \sup_{\|y^*\|\leq 1} \sup_{\|x\|\leq 1} \|\langle Tx, y^*\rangle e_1\|_\Lambda = \|T\|.$$

Thus we define an isometry $T \mapsto \phi_T$ from $\mathcal{L}(X,Y)$ into $\mathcal{L}(\mathcal{L}(Y,\Lambda), \mathcal{L}(X,\Lambda))$. It is then clear that $T \in \mathcal{A}_\Lambda(X,Y)$ if and only if $\phi_T(\mathcal{L}(Y,\Lambda)) \subseteq \mathcal{A}(X,\Lambda)$.

Now let $T \in \mathcal{A}_\Lambda(X,Y)$ be given. By the above discussion ϕ_T is a linear operator from $\mathcal{L}(Y,\Lambda)$ into $\mathcal{A}(X,\Lambda)$.
We show that ϕ_T has closed graph:
Suppose $S_n \xrightarrow[\infty]{n} S$ in $\mathcal{L}(Y,\Lambda)$ and $\phi_T(S_n) = S_nT \xrightarrow[\infty]{n} R$ in $(\mathcal{A}(X,\Lambda), \alpha(\cdot))$. Then, since

$$\|S_nT - R\| \leq \alpha(S_nT - R) \xrightarrow[\infty]{n} 0,$$

it also follows that $S_n(Tx) \xrightarrow[\infty]{n} Rx$ in Λ for all $x \in X$. However, we also have

$$\|S_n(Tx) - S(Tx)\| \leq \|S_n - S\|\|Tx\| \xrightarrow[\infty]{n} 0$$

for all $x \in X$. Therefore, $R = ST = \phi_T(S)$, i.e $\phi_T(S_n) \xrightarrow[\infty]{n} \phi_T(S)$ in $(\mathcal{A}(X,\Lambda), \alpha(\cdot))$. The underlying spaces being Banach spaces, the Closed Graph Theorem assures us that $\phi_T : \mathcal{L}(Y,\Lambda) \to (\mathcal{A}(X,\Lambda), \alpha(\cdot))$ is bounded. Thus, we may define

$$\alpha_\Lambda(T) = \sup\{\alpha(ST) : S \in \mathcal{L}(Y,\Lambda), \|S\| \leq 1\}.$$

Then $\alpha_\Lambda(\cdot)$ defines a norm on $\mathcal{A}_\Lambda(X,Y)$. For example if $\alpha_\Lambda(T) = 0$, then

$$\begin{aligned}
\|Tx\| &= \sup_{\|y^*\|\leq 1} |\langle Tx, y^*\rangle| = \sup_{\|y^*\|\leq 1} \|(y^* \otimes e_1)(Tx)\|_\Lambda \\
&\leq \sup\{\alpha(ST) : S \in \mathcal{L}(Y,\Lambda), \|S\| \leq 1\}\|x\| \\
&= \alpha_\Lambda(T)\|x\| = 0
\end{aligned}$$

for all $x \in X$, i.e. $T = 0$. Moreover, $\|T\| \leq \alpha_\Lambda(T)$ for all $T \in \mathcal{A}_\Lambda(X,Y)$. Since this is true for all Banach spaces X and Y, it therefore follows that:

Proposition 5.3.1 $(\mathcal{A}_\Lambda, \alpha_\Lambda(\cdot))$ *is a normed operator ideal.*

Clearly, $\alpha_\Lambda(T)$ is the operator norm of ϕ_T considered as an element of $\mathcal{L}(\mathcal{L}(Y,\Lambda),\mathcal{A}(X,Y))$. This proves that the mapping $T \mapsto \phi_T$ defines an isometry from $\mathcal{A}_\Lambda(X,Y)$ into $\mathcal{L}(\mathcal{L}(Y,\Lambda),\mathcal{A}(X,Y))$. Thus, $\mathcal{A}_\Lambda(X,Y)$ is isometrically associated with a subspace $A_\Lambda := \{\phi_T : T \in \mathcal{A}_\Lambda(X,Y)\}$ of the Banach space $\mathcal{L}(\mathcal{L}(Y,\Lambda),\mathcal{A}(X,Y))$.

Taking $R \in \mathcal{L}(\mathcal{L}(Y,\Lambda),\mathcal{A}(X,Y))$ from the closure of the subspace A_Λ in the operator norm, let $(\phi_{T_n}) \in A_\Lambda$ so that $\phi_{T_n} \underset{\infty}{\overset{n}{\longrightarrow}} R$ in the operator norm of $\mathcal{L}(\mathcal{L}(Y,\Lambda),\mathcal{A}(X,Y))$. Since uniform convergence of operators implies strong operator convergence, it follows that $\phi_{T_n}(S) \underset{\infty}{\overset{n}{\longrightarrow}} R(S)$ in $\mathcal{A}(X,Y)$ for all $S \in \mathcal{L}(Y,\Lambda)$. This implies that

$$\|ST_n - R(S)\| \leq \alpha(ST_n - R(S)) \underset{\infty}{\overset{n}{\longrightarrow}} 0,$$

so that for each $x \in X$ and each $S \in \mathcal{L}(Y,\Lambda)$, we also have $ST_n x \underset{\infty}{\overset{n}{\longrightarrow}} R(S)x$. Denote the restriction of the operator norm of $\mathcal{L}(\mathcal{L}(Y,\Lambda),\mathcal{A}(X,Y))$ to the subspace A_Λ by $\|\cdot\|_{A_\Lambda}$. From the isometry $T \mapsto \phi_T$ discussed above, we then conclude that

$$\|T_n - T_m\| \leq \alpha_\Lambda(T_n - T_m) = \|\phi_{T_n} - \phi_{T_m}\|_{A_\Lambda} \to 0 \text{ as } m,n \to \infty.$$

Let $T \in \mathcal{L}(X,Y)$ so that $T_n \underset{\infty}{\overset{n}{\longrightarrow}} T$ in the operator norm of $\mathcal{L}(X,Y)$. Then, for each $x \in X$ we have $Tx = \lim_n T_n x$. Therefore, we conclude that

$$\|ST_n x - STx\| \leq \|S\|\|T_n x - Tx\| \underset{\infty}{\overset{n}{\longrightarrow}} 0,$$

so that we have $R(S)x = STx$ for each $x \in X$ and each $S \in \mathcal{L}(Y,\Lambda)$, i.e. $R = \phi_T$. Thereby we have shown that for each pair of Banach spaces X and Y, the vector space A_Λ is a closed subspace of the complete space $\mathcal{L}(\mathcal{L}(Y,\Lambda),\mathcal{A}(X,Y))$ and thus that:

Theorem 5.3.2 $(\mathcal{A}_\Lambda, \alpha_\Lambda(\cdot))$ *is a Banach operator ideal.*

It is clear from the definition that $\mathcal{A}_\Lambda(X,\Lambda) = \mathcal{A}(X,\Lambda)$ for all Banach spaces X. Recall that an operator ideal $\mathcal{A}$ is said to be *surjective* if for all Banach spaces X, Y, Z, and each $T \in \mathcal{L}(X,Y)$ for which there exists a surjective operator (quotient map) $Q \in \mathcal{L}(Z,X)$ so that $TQ \in \mathcal{A}(Z,Y)$, it follows that $T \in \mathcal{A}(X,Y)$. It is easily seen that if $\mathcal{A}$ is a surjective ideal, then so is $\mathcal{A}_\Lambda$.

For a given Banach operator ideal $(\mathcal{A}, \alpha)$, consider the associated operator ideal $(\mathcal{A}^{\diamondsuit}_{\Lambda}, \alpha^{\diamondsuit}_{\Lambda})$, whereby we let

$$\begin{aligned}\mathcal{A}^{\diamondsuit}_{\Lambda}(X,Y) &= \{T \in \mathcal{L}(X,Y) : TS \in \mathcal{A}(\Lambda, Y), \forall S \in B(\Lambda, X)\} \\ \alpha^{\diamondsuit}_{\Lambda}(T) &= \sup\{\alpha(TS) : S \in \mathcal{L}(\Lambda, X), \|S\| \leq 1\}.\end{aligned}$$

In this case, we have $\|Tx\| = \|(T \circ e_1 \otimes x)e_1\|$ for $x \in X$. Therefore,

$$\|T\| \leq \sup_{\|x\|\leq 1} \|T \circ (e_1 \otimes x)\| \leq \sup\{\alpha(TS) : S \in \mathcal{L}(\Lambda, X), \|S\| \leq 1\} = \alpha^{\diamondsuit}_{\Lambda}(T).$$

It is clear from the definition that $\mathcal{A}^{\diamondsuit}_{\Lambda}(\Lambda, Y) = \mathcal{A}(\Lambda, Y)$ for all Banach spaces Y. Recall that an operator ideal $\mathcal{A}$ is said to be *injective* if for all Banach spaces X, Y, Y_0 such that Y is isometrically embedded into Y_0 by $J \in \mathcal{L}(Y, Y_0)$, it follows from $T \in \mathcal{L}(X, Y)$ and $JT \in \mathcal{A}(X, Y_0)$ that $T \in \mathcal{A}(X, Y)$. It is easily seen that if $\mathcal{A}$ is an injective ideal, then so is $\mathcal{A}^{\diamondsuit}_{\Lambda}$.

The operator ideal $\bar{\mathcal{F}}$, consisting of the closures of the components $\mathcal{F}(X, Y)$ in $\mathcal{L}(X, Y)$ (in operator norm), is called the ideal of approximable operators.

Recall that an operator $T \in \mathcal{L}(X, Y)$ belongs to the component $\mathcal{A}^{max}(X, Y)$ of the *maximal hull* $\mathcal{A}^{max}$ of an ideal $\mathcal{A}$ if $RTS \in \mathcal{A}(X_0, Y_0)$ for all $S \in \bar{\mathcal{F}}(X_0, X), \forall R \in \bar{\mathcal{F}}(Y, Y_0)$ and for all Banach spaces X_0, Y_0. The ideal $\mathcal{A}$ is called *maximal* if $\mathcal{A} = \mathcal{A}^{max}$, i.e if $\mathcal{A}(X, Y) = \mathcal{A}^{max}(X, Y)$ for all Banach spaces X, Y. A Banach operator ideal $(\mathcal{A}, \alpha)$ is a maximal Banach ideal if $(\mathcal{A}, \alpha) = (\mathcal{A}^{max}, \alpha^{max})$ (isometrically). Here, the ideal norm α^{max} is defined by

$$\alpha^{max}(T) := \sup \alpha(RTS),$$

where the supremum is taken over all $S \in \bar{\mathcal{F}}(X_0, X)$, $R \in \bar{\mathcal{F}}(Y, Y_0)$, $\|S\| \leq 1, \|R\| \leq 1$ and all Banach spaces X_0, Y_0. If the Banach ideal $(\mathcal{A}, \alpha)$ satisfies $\mathcal{A} = \mathcal{A}^{max}$, then the components $(\mathcal{A}(X, Y), \alpha(\cdot))$ and $(\mathcal{A}(X, Y), \alpha^{max}(\cdot))$ are both Banach spaces for which $\alpha^{max}(T) \leq \alpha(T)$ for all $T \in \mathcal{A}(X, Y)$. Thus the norms should be equivalent (at least) on each component $\mathcal{A}(X, Y)$.

Let the scalar sequence space Λ be a normal BK-space with AK, i.e. Λ is a Banach sequence space such that the coordinate projections $(\lambda_i) \mapsto \lambda_j$ are continuous for all j and the standard basis $\{e_n : n \in \mathbb{N}\}$ is a Schauder basis for Λ. Here "normal" means that it follows from $|\alpha_i| \leq |\lambda_i|$ for all i and $(\lambda_i) \in \Lambda$ that $(\alpha_i) \in \Lambda$. If, moreover, the dual space Λ^* of the normal BK-space with AK is again a normal BK-space with the AK-property, then we call Λ a DAK-space. We refer to the paper [36] for results on the operator ideal of so called Λ-compact operators. An operator $T : X \to Y$ is said to be Λ-compact if there exist operators $P \in K(X, \Lambda)$ and $Q \in K(\Lambda, Y)$ such that $T = QP$. The family of all Λ-compact operators from X to Y is denoted by $K_\Lambda(X, Y)$. If we let

$$\kappa(T) = \inf\{\|Q\|\|P\| : T = QP\},$$

then (K_Λ, κ) is a quasinormed operator ideal. Clearly, for each pair X, Y of Banach spaces we have $K_\Lambda(X, Y) \subseteq K(X, Y)$. It is shown in [36] that $K_\Lambda(X, \Lambda) = K(X, \Lambda)$ for all Banach spaces X and that $\kappa(T) = \|T\|$ for all $T \in K(X, \Lambda)$. Similarly, when Λ is a DAK-space, then $K_\Lambda(\Lambda, X) = K(\Lambda, X)$ for all Banach spaces X and $\kappa(T) = \|T\|$ for all $T \in K(\Lambda, X)$. Recall that if X^* or Y has the approximation property, then each compact operator $X \to Y$ is approximable, i.e. $K(X, Y) = \bar{\mathcal{F}}(X, Y)$ in this case. By our assumptions on Λ, it thus follows that $K(X, \Lambda) = \bar{\mathcal{F}}(X, \Lambda)$ for all X and that if Λ is a DAK-space, then $K(\Lambda, X) = \bar{\mathcal{F}}(\Lambda, X)$ for all X.

Theorem 5.3.3 *Let Λ be a DAK-space. If $(\mathcal{A}, \alpha)$ is a maximal Banach operator ideal, then $\mathcal{A}_\Lambda$ is a maximal operator ideal, i.e. $\mathcal{A}_\Lambda(X, Y) = \mathcal{A}_\Lambda^{max}(X, Y)$ for all Banach spaces X, Y. Moreover, the norms $\alpha_\Lambda(\cdot)$ and $\alpha_\Lambda^{max}(\cdot)$ are equivalent on each component $\mathcal{A}_\Lambda(X, Y)$.*

Proof Let $T \in \mathcal{A}_\Lambda^{max}(X, Y)$. To show that $T \in \mathcal{A}_\Lambda(X, Y)$, we consider any $S \in \mathcal{L}(Y, \Lambda)$ and prove that $ST \in \mathcal{A}(X, \Lambda)$: Since $(\mathcal{A}, \alpha)$ is maximal, it is enough to prove that $ST \in \mathcal{A}^{max}(X, \Lambda)$. Therefore, let $P \in \bar{\mathcal{F}}(X_0, X)$ and $Q \in \bar{\mathcal{F}}(\Lambda, Y_0)$ for arbitrary Banach spaces X_0 and Y_0. Since $Q \in K_\Lambda(\Lambda, Y_0)$, we have $Q = VW$, where $W \in \bar{\mathcal{F}}(\Lambda, \Lambda)$ and $V \in \bar{\mathcal{F}}(\Lambda, Y_0)$. By the operator ideal property, we have $ST \in \mathcal{A}_\Lambda^{max}(X, \Lambda)$ and so

$$WSTP \in \mathcal{A}_\Lambda(X_0, \Lambda) = \mathcal{A}(X_0, \Lambda).$$

Thus, we have

$$Q(ST)P = VWSTP \in \mathcal{A}(X_0, Y_0),$$

where the operators P and Q are arbitrarily chosen. This proves that $ST \in \mathcal{A}^{max}(X, \Lambda)$.

The above argument shows that $\mathcal{A}_\Lambda^{max}(X, Y) = \mathcal{A}_\Lambda(X, Y)$ for every pair X, Y of Banach spaces.

Since by definition $\alpha_\Lambda^{max} \leq \alpha_\Lambda$ on each Banach space $\mathcal{A}_\Lambda(X, Y)$, the norms have to be equivalent on each component $\mathcal{A}_\Lambda(X, Y)$. □

Recall that an operator ideal $\mathcal{A}$ is called *regular* if it follows from $T \in \mathcal{L}(X,Y)$ and $C_Y T \in \mathcal{A}(X, Y^{**})$ (where $C_Y : Y \hookrightarrow Y^{**}$ is the canonical embedding) that $T \in \mathcal{A}(X,Y)$. Every maximal operator ideal is regular (cf. [54], 4.9.11, page 77). From Theorem 5.3.3 it therefore follows that if $(\mathcal{A}, \alpha)$ is a maximal Banach operator ideal, then $\mathcal{A}_\Lambda$ is regular. The operator ideal (Π_p, π_p) (for $1 \leq p < \infty$) is a maximal ideal (cf. [30], Remark 6.18, page 140). We may thus conclude from Theorem 5.3.3 that:

Corollary 5.3.4 *Let $1 < p < \infty$. The operator ideal Lt_p is maximal, hence regular.*

Using results about tensor norms and the fact that a maximal Banach operator ideal is associated with a finitely generated tensor norm, it follows that:

Theorem 5.3.5 *(cf. Corollary 4 in [25], Chapter II, §17.8). If $(\mathcal{A}, \alpha)$ is a maximal Banach operator ideal, then $T \in \mathcal{A}(X,Y)$ if and only if $T^{**} \in \mathcal{A}(X^{**}, Y^{**})$ for all Banach spaces X, Y. In this case $\alpha(T) = \alpha(T^{**})$.*

Theorem 5.3.6 *Let Λ be a reflexive BK-space with AK and suppose $(\mathcal{A}, \alpha)$ is a maximal Banach operator ideal. Then $T \in \mathcal{A}_\Lambda(X,Y)$ if and only if $T^{**} \in \mathcal{A}_\Lambda(X^{**}, Y^{**})$. In this case $\alpha_\Lambda^{max}(T) = \alpha_\Lambda^{max}(T^{**})$ and $\alpha_\Lambda(T) \leq \alpha_\Lambda(T^{**})$.*

Proof Suppose $T^{**} \in \mathcal{A}_\Lambda(X^{**}, Y^{**})$. Since for each $S \in \mathcal{L}(Y, \Lambda)$ we have

$$(ST)^{**} = S^{**}T^{**} \in \mathcal{A}(X^{**}, \Lambda)$$

and $(\mathcal{A}, \alpha)$ is a maximal Banach ideal, it follows that $ST \in \mathcal{A}(X, \Lambda)$ and

$$\alpha(ST) = \alpha((ST)^{**}) = \alpha(S^{**}T^{**}).$$

This proves that $T \in \mathcal{A}_\Lambda(X,Y)$ and $\alpha_\Lambda(T) \leq \alpha_\Lambda(T^{**})$.

Conversely, Λ is a DAK-space, so that by Theorem 5.3.3, we have $\mathcal{A}_\Lambda(X,Y) = \mathcal{A}_\Lambda^{max}(X,Y)$, for all Banach spaces X, Y. Therefore, $(\mathcal{A}_\Lambda, \alpha_\Lambda^{max})$ is a maximal Banach ideal. Thus, if $T \in \mathcal{A}_\Lambda(X,Y)$, then by Theorem 5.3.5 we have $T^{**} \in \mathcal{A}_\Lambda(X^{**}, Y^{**})$.

The norm equality also follows from Theorem 5.3.5. □

The maximality of the Banach ideal (Π_p, π_p) and Theorem 5.3.6 now implies that a much stronger version of our Proposition 5.2.3 is true:

Corollary 5.3.7 *Let $1 < p < \infty$. A bounded linear operator $T : X \to Y$ is sequentially p-limited if and only if $T^{**} : X^{**} \to Y^{**}$ is sequentially p-limited.*

Given a Banach operator ideal $(\mathcal{A}, \alpha)$, we recall that a Banach operator ideal $(\mathcal{A}^d, \alpha^d)$, called the *dual ideal* of $(\mathcal{A}, \alpha)$, is defined by the components

$$\mathcal{A}^d(X, Y) := \{T \in \mathcal{L}(X, Y) : T^* \in \mathcal{A}(Y^*, X^*)\},$$

where X, Y run through the family of all Banach spaces. Here,

$$\alpha^d(T) = \alpha(T^*).$$

Let $1 < p < \infty$. Since each $T \in \mathcal{L}(\ell_{p'}, X^*)$ is weak-to-weak continuous and $\ell_{p'}$ is reflexive, we have:

Lemma 5.3.8 *Let $1 < p < \infty$. Each $T \in \mathcal{L}(\ell_{p'}, X^*)$ is weak*-to-weak* continuous.*

Consider the case when $\Lambda = \ell_p$ with $1 < p < \infty$. In this case denote the Banach operator ideal $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ (respectively, $(\mathcal{A}_\Lambda^\diamondsuit, \alpha_\Lambda^\diamondsuit)$) by $(\mathcal{A}_p, \alpha_p)$ (respectively, $(\mathcal{A}_p^\diamondsuit, \alpha_p^\diamondsuit)$). Using Lemma 5.3.8 to realise that $S \in \mathcal{L}(\ell_{p'}, Y^*)$ if and only if there exists $R \in \mathcal{L}(Y, \ell_p)$ such that $R^* = S$, one verifies easily that:

Proposition 5.3.9 *Let $1 < p < \infty$. Then $T \in (\mathcal{A}_{p'}^\diamondsuit)^d(X, Y)$ if and only if $T \in (\mathcal{A}^d)_p(X, Y)$; in this case $(\alpha_{p'}^\diamondsuit)^d(T) = (\alpha^d)_p(T)$.*

It follows from Proposition 5.3.9 that:

Corollary 5.3.10 *For $1 < p < \infty$, we have*

$$(\Pi_p^d)_p(X, Y) := \{T \in \mathcal{L}(X, Y) : T^* S \in \Pi_p(\ell_{p'}, X^*), \forall S \in \mathcal{L}(\ell_{p'}, Y^*)\}.$$

Remark 5.3.11 *In our discussion above it is clear that to obtain the necessary isometric embedding $T \mapsto \phi_T$, we need to assume the properties on Λ stated at the beginning of this section (in particular, that $\|e_n\|_\Lambda = 1$ for all n). These properties are for instance also shared by some Orlicz sequence spaces. For example, if for $1 \leq p < \infty$ we let*

$$N_p(t) = \begin{cases} t^p(1 + |\ln t|), & \textit{for } t > 0; \\ 0, & \textit{if } t = 0, \end{cases}$$

then N_p defines an Orlicz function. The norm on the corresponding Orlicz sequence space

$$h_{N_p} := \left\{ (\alpha_i) : \sum_{n=1}^{\infty} N_p\left(\frac{|\alpha_n|}{\rho}\right) < \infty, \forall \rho > 0 \right\},$$

is given by

$$\|(\alpha_i)\|_{N_p} := \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} N_p\left(\frac{|\alpha_n|}{\rho}\right) \leq 1 \right\}.$$

The set $\{e_n\}_{n=1}^{\infty}$ of unit vectors is a symmetric basis for h_{N_p} and since $N_p(t) > 1$ if $t > 1$, $N_p(1) = 1$ and $\lim_{t \downarrow 1} N_p(t) = 1$, it follows that $\|e_n\|_{N_p} = 1$ for all $n \in \mathbb{N}$.

Our result Theorem 5.2.2 also follows from the (general) operator ideal approach discussed in this section (in particular from Theorem 5.3.2): Take $(\mathcal{A}, \alpha) = (\Pi_p, \pi_p)$ and $\Lambda = \ell_p$ (where $1 \leq p < \infty$) to verify that in this case $(\mathcal{A}_\Lambda, \alpha_\Lambda) = (Lt_p, \ell t_p)$ and hence that the space $(Lt_p(X,Y), \ell t_p(\cdot))$ of sequentially p-limited operators is a Banach space.

5.4 More classes of operators

Recall from the book [30] (Chapter 10, page 197) that an operator $T : X \to Y$ is called (q,p)-summing (with $1 \leq p, q < \infty$) if there is an induced operator

$$\widehat{T} : \ell_p^{weak}(X) \to \ell_q^{strong}(Y) : (x_n) \mapsto (Tx_n).$$

The vector space of (q,p)-summing operators is denoted by $\Pi_{q,p}(X,Y)$; it is normed by the norm

$$\pi_{q,p}(T) = \|\widehat{T}\|,$$

where $\|\widehat{T}\|$ denotes the operator norm of $\widehat{T}$. A bounded linear operator $T \in \mathcal{L}(X,Y)$ is (q,p)-summing if and only if there is some $C \geq 0$ for which

$$(*) \qquad \left(\sum_{i=1}^{n} \|Tx_i\|^q\right)^{1/q} \leq C \sup_{x^* \in U_{X^*}} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p\right)^{1/p},$$

no matter how the finite set $\{x_1, \ldots, x_n\}$ of vectors from X is chosen. Moreover, $\pi_{q,p}(T)$ is the least such constant C. Using the version $(*)$ of the definition of a (q,p)-summing operator, one soon verifies that only the zero operator can be (q,p)-summing if $q < p$. It will therefore be natural to assume that $p \leq q$. Under the assumption $1 \leq p \leq q < \infty$, the pair $(\Pi_{q,p}, \pi_{q,p})$ is an injective Banach ideal (cf. [30], Proposition 10.2). Observe that an operator $T \in \mathcal{L}(X,Y)$ is (p,p)-summing if and only if it is p-summing (refer to [30], page 31 for the definition of p-summing operator in terms of the corresponding version of the inequality $(*)$ for the case $p = q$).

Let $1 \leq p \leq q < \infty$ and let $1 \leq r < \infty$. If we let $(\mathcal{A}, \alpha) = (\Pi_{q,p}, \pi_{q,p})$ in our general discussion of Section 3, then:

1. We denote $(\mathcal{A}_{\ell_r}, \alpha_{\ell_r})$ by $(Lt_{q,p,r}, \ell t_{q,p,r})$. In this case we have $T \in Lt_{q,p,r}(X,Y)$ if and only if $ST \in \Pi_{q,p}(X, \ell_r)$ for all $S \in \mathcal{L}(Y, \ell_r)$, i.e. if and only if

$$\sum_{n=1}^{\infty} \|STx_n\|_r^q < \infty, \quad \forall\, S \in \mathcal{L}(Y, \ell_r),\, \forall\, (x_n) \in \ell_p^{weak}(X).$$

   Also, for $T \in Lt_{q,p,r}(X,Y)$, we have

$$\ell t_{q,p,r}(T) = \sup_{S \in U_{\mathcal{L}(Y,\ell_r)}} \pi_{q,p}(ST).$$

2. In case of $p = q = r$, we clearly have $(Lt_{p,p,p}, \ell t_{p,p,p}) = (Lt_p, \ell t_p)$.

3. In case of $p = q$, we denote $(Lt_{q,p,r}, \ell t_{q,p,r})$ by $(Lt_{p,r}, \ell t_{p,r})$. In this case we have $T \in Lt_{p,r}(X,Y)$ if and only if $ST \in \Pi_p(X, \ell_r)$ for all $S \in \mathcal{L}(Y, \ell_r)$. The operators $T \in Lt_{p,r}(X,Y)$ will be called sequentially (p,r)-limited. In case of $p = r$, we again have $(Lt_{p,p}, \ell t_{p,p}) = (Lt_p, \ell t_p)$

By Theorem 5.3.2, the pairs $(Lt_{q,p,r}, \ell t_{q,p,r})$ and $(Lt_{p,r}, \ell t_{p,r})$ are Banach operator ideals.

Using Theorem 2.8 in [30], we have the following inclusion result:

Theorem 5.4.1 *Let $1 \leq p \leq q < \infty$ and $1 \leq r < \infty$. Then $Lt_{p,r}(X,Y) \subseteq Lt_{q,r}(X,Y)$. Moreover, for $T \in Lt_{p,r}(X,Y)$ we have $\ell t_{q,r}(T) \leq \ell t_{p,r}(T)$.*

Proof Given $T \in Lt_{p,r}(X,Y)$ and $S \in \mathcal{L}(Y, \ell_r)$, it follows that $ST \in \Pi_p(X, \ell_r)$ and $\pi_p(ST) \leq \|S\| \ell t_{p,r}(T)$. By Theorem 2.8 in [30] we therefore have $ST \in \Pi_q(X, \ell_r)$ and

$$\pi_q(ST) \leq \pi_p(ST) \leq \|S\| \ell t_{p,r}(T).$$

Since S was arbitrary, it follows that $T \in Lt_{q,r}(X,Y)$ and

$$\ell t_{q,r}(T) = \sup_{S \in U_{\mathcal{L}(Y,\ell_r)}} \pi_q(ST) \leq \sup_{S \in U_{\mathcal{L}(Y,\ell_r)}} \|S\| \ell t_{p,r}(T) = \ell t_{p,r}(T).$$

□

Generalizing Theorem 5.4.1, we may use Theorem 10.4 in [30] in a similar fashion to prove that:

Theorem 5.4.2 *Let $1 \leq t < \infty$ and suppose that $1 \leq p_j \leq q_j < \infty$ ($j = 1, 2$) satisfy $p_1 \leq p_2, q_1 \leq q_2$ and*

$$\frac{1}{p_1} - \frac{1}{q_1} \leq \frac{1}{p_2} - \frac{1}{q_2}.$$

Then

$$Lt_{q_1,p_1,r}(X,Y) \subseteq Lt_{q_2,p_2,r}(X,Y)$$

and for each $T \in Lt_{q_1,p_1,r}(X,Y)$ we have

$$\ell t_{q_2,p_2,r}(T) \leq \ell t_{q_1,p_1,r}(T).$$

Using two more results from [30] (namely, Lemma 2.23 and Theorem 2.22) we obtain the following multiplication theorem:

Theorem 5.4.3 *Let $1 \leq p, q, r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $T \in \Pi_q(X,Y)$ and $S \in Lt_{p,r}(Y,Z)$. Then $ST \in Lt_r(X,Z)$ and*

$$\ell t_r(ST) \leq \ell t_{p,r}(S)\, \pi_q(T).$$

Proof Let $(x_i) \in \ell_r^{weak}(X)$ and put $\gamma := \|(x_m)\|_r^{weak}$. By Lemma 2.23 of [30], let

$$Tx_n = \sigma_n y_n, \ \forall n \in \mathbb{N},$$

where $(\sigma_n) \in \ell_q$ and $(y_n) \in \ell_p^{weak}(Y)$ such that

$$\|(\sigma_n)\|_q \leq \gamma^{r/q} \text{ and } \|(y_n)\|_p^{weak} \leq \gamma^{r/p}\, \pi_q(T).$$

For $R \in \mathcal{L}(Z, \ell_r)$, with $\|R\| \le 1$, we have $(RSy_n) \in \ell_p^s(\ell_r)$. It follows that:

$$\begin{aligned}
\left(\sum_n \|RSTx_n\|_r^r\right)^{1/r} &= \left(\sum_n |\sigma_n|^r \|RSy_n\|_r^r\right)^{1/r} \\
&\le \left(\sum_n |\sigma_n|^q\right)^{1/q} \left(\sum_n \|RSy_n\|_r^p\right)^{1/p} \\
&\le \gamma^{r/q} \ell t_{p,r}(S) \|(y_n)\|_p^{weak} \\
&\le \gamma^{r/q} \ell t_{p,r}(S) \gamma^{r/p} \pi_q(T) \\
&= \ell t_{p,r}(S)\, \pi_q(T)\, \|(x_n)\|_r^{weak}.
\end{aligned}$$

This shows that $ST \in Lt_r(X, Z)$. Taking firstly the supremum over all $\|(x_n)\|_r^{weak} \le 1$ and then the supremum over all $R \in \mathcal{L}(Z, \ell_r)$, with $\|R\| \le 1$, it also follows that $\ell t_r(ST) \le \ell t_{p,r}(S)\, \pi_q(T)$. □

Corollary 5.4.4 *Let $1 \le p, q < \infty$ be such that $1 \le \frac{1}{p} + \frac{1}{q}$. If $S \in Lt_{p,1}(Y, Z)$ and $T \in \Pi_q(X, Y)$, then $ST \in Lt_1(X, Z)$ and*

$$\ell t_1(ST) \le \ell t_{p,1}(S)\, \pi_q(T).$$

Proof For $p = 1$ we have $S \in Lt_1(Y, Z)$ and so by the operator ideal properties we also have $ST \in Lt_1(X, Z)$ and

$$\begin{aligned}
\ell t_1(ST) &= \sup_{R \in U_{\mathcal{L}(Z,\ell_1)}} \pi_1(RST) \\
&\le \sup_{R \in U_{\mathcal{L}(Z,\ell_1)}} \pi_1(RS)\|T\| \le \ell t_{1,1}(S) \pi_q(T).
\end{aligned}$$

Now, assume $p > 1$. Then $1 \le q \le p' < \infty$, hence $T \in \Pi_{p'}(X, Y)$ and $\pi_{p'}(T) \le \pi_q(T)$ by Theorem 2.8 in [30]. The result follows by application of Theorem 5.4.3. □

Let $1 \le q < \infty$. Recall from Proposition 5.23 in [30] (page 112) that $T \in \mathcal{L}(X, Y)$ is q-nuclear if and only if it has a representation

$$T = \sum_{i=1}^{\infty} x_i^* \otimes y_i,$$

where $(x_i^*) \in \ell_q^{strong}(X^*)$ and $(y_i) \in \ell_{q'}^{weak}(Y)$. The norm on the space $\mathcal{N}_q(X,Y)$ of q-nuclear operators is then given by

$$\nu_q(T) := \inf\{\|(x_i^*)\|_q^{strong}\|(y_i)\|_{q'}^{weak} : T = \sum_{i=1}^{\infty} x_i^* \otimes y_i\}.$$

From the notation of our earlier discussion in this chapter, it follows that if $\mathcal{A} = \mathcal{N}_q$ (and $\alpha = \nu_q$) and $\Lambda = \ell_r$, then $\mathcal{A}_\Lambda$ becomes $(\mathcal{N}_q)_r$ and $\alpha_\Lambda = (\nu_q)_r$. With this notation in mind, we have the following composition result:

Theorem 5.4.5 *Let $1 \le p, q, r < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Let $T \in \mathcal{N}_q(X,Y)$ and $S \in Lt_{p,r}(Y,Z)$. Then $ST \in (\mathcal{N}_r)_r(X,Z)$ and*

$$(\nu_r)_r(ST) \le \ell t_{p,r}(S)\, \nu_q(T).$$

Proof Let $T \in \mathcal{N}_q(X,Y)$. For $\delta > 0$ being arbitrarily given, let $T = \sum\limits_{i=1}^{\infty} x_i^* \otimes y_i$, with $(x_i^*) \in \ell_q^{strong}(X^*)$ and $(y_i) \in \ell_{q'}^{weak}(Y)$ so chosen that

$$\|(x_i^*)\|_q^{strong} \le 1 \text{ and } \|(y_i)\|_{q'}^{weak} \le \nu_q(T) + \delta\,.$$

From $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, we get $\frac{1}{q'} = \frac{1}{p} + \frac{1}{r'}$. Let $R \in \mathcal{L}(Z, \ell_r)$; then $RS \in \Pi_p(Y, \ell_r)$. Apply Lemma 2.23 (in [30]) in the current setting to obtain $(\sigma_n) \in \ell_p$ and $(z_n) \in \ell_{r'}^{weak}(\ell_r)$ so that if $\gamma = \|(y_i)\|_{q'}^{weak}$, then

$$\|(\sigma_n)\|_p \le \gamma^{q'/p},\ \|(z_n)\|_{r'}^{weak} \le \gamma^{q'/r'}\pi_p(RS) \text{ and } RSy_n = \sigma_n z_n,\ \forall n.$$

Then we have

$$RST = \sum_{i=1}^{\infty} \sigma_i x_i^* \otimes z_i, \text{ where } (\sigma_i x_i^*) \in \ell_r^{strong}(X^*).$$

Thus, $RST \in \mathcal{N}_r(X, \ell_r)$. Since this holds for all $R \in \mathcal{L}(Z, \ell_r)$, it follows that $ST \in (\mathcal{N}_r)_r(X,Z)$. Also, if we assume that $\|R\| \le 1$ in the above calculation, then

$$\begin{aligned} \nu_r(RST) &\le \|(\sigma_i x_i^*)\|_r^s\, \|(z_i)\|_{r'}^{weak} \\ &\le \|(\sigma_i)\|_p\, \gamma^{q'/r'}\pi_p(RS) \\ &\le \gamma^{q'/p}\gamma^{q'/r'}\pi_p(RS) \\ &= \|(y_i)\|_{q'}^{weak}\pi_p(RS) \\ &\le (\nu_q(T)+\delta)\pi_p(RS). \end{aligned}$$

$R \in \mathcal{L}(Z, \ell_r)$ (with $\|R\| \le 1$) and $\delta > 0$ being arbitrarily chosen, it follows that $(\nu_r)_r(ST) \le \nu_q(T)\ell t_{p,r}(S)$. $\square$

Chapter 6

On Dunford-Pettis-type functions

6.1 Polynomials

We follow the conventions used by Dineen (cf. [31], pages 2-16) and use the following notation:

Notation 6.1.1 *Let X and Y be vector spaces over $\mathbb{C}$. Then:*

(a) We denote by $x^n y^m$ the element $(x, x, \ldots, x, y, y, \ldots, y)$ of $X^n \times Y^m$, where $x \in X$ and $y \in Y$.

(b) We denote by $\mathcal{L}_a(^nX; Y)$ the space of all n-linear mappings from X^n to Y, i.e. $\mathcal{L}_a(^nX; Y) = \{f : X^n \to Y : f \text{ is } n\text{-linear}\}$.

(i) When $n = 1$: $f \in \mathcal{L}_a(X; Y)$ is linear.

(ii) When $n = 2$: $\mathcal{B}_a(X; Y) = \mathcal{L}_a(^2X; Y)$ is the space of 2-linear (or bilinear) functions from X^2 into Y.

(c) When $Y = \mathbb{C}$, we let $\mathcal{L}_a(^nX) = \mathcal{L}_a(^nX; \mathbb{C})$ and $\mathcal{L}_a(X) = X'$. Also, we let $\mathcal{L}_a(^0X,\ Y)$ be the set of constant mappings from X into Y.
If α is a seminorm on Y and $A \subseteq \mathcal{D}(f)$, we let

$$\|f\|_{\alpha,A} := \sup_{x \in A} \alpha(f(x)).$$

Definition 6.1.2 *The diagonal mapping $\Delta_n : X \to X^n$ is defined by $\Delta_n x = (x, x, \ldots, x)$.*

Definition 6.1.3 *Let X,Y be vector spaces over $\mathbb{C}$. A mapping $P : X \to Y$ is called an n-homogeneous polynomial from X to Y if there exists an element $L \in \mathcal{L}_a({}^nX; Y)$ such that $P = L \circ \Delta_n$; that is we have $P(x) = L(x, x, \ldots, x)$ for all $x \in X$.*

Definition 6.1.4 *An n-linear mapping L from X^n into Y is said to be symmetric if*

$$L(x_1, x_2, \ldots, x_n) = L(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}),$$

for any $(x_1, x_2, \ldots, x_n) \in X^n$ and any permutation σ of $\{1, 2, \ldots, n\}$.

Notation 6.1.5
(a) We let $\mathcal{P}_a({}^nX; Y)$ be the vector space of all n-homogeneous polynomials from X to Y. A polynomial from X into Y is any finite sum of homogeneous polynomials from X into Y. The vector space of all polynomials from X into Y is denoted by $\mathcal{P}_a(X; Y)$.

(b) We let $\mathcal{L}_a^s({}^nX; Y)$ be the vector space of all symmetric n-linear mappings from X into Y.

Remark 6.1.6 *Given a $L \in \mathcal{L}_a({}^nX; Y)$, we associate with L a symmetric n-linear mapping by*

$$s(L)(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} L(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

where S_n is the set of all permutations of the first n natural numbers.

The following properties are easily verified (cf. [31], page 6):

(a) $s(L) \in \mathcal{L}_a^s({}^nX; Y)$ for all $L \in \mathcal{L}_a({}^nX; Y)$.

(b) $s(L) = L$ if and only if $L \in \mathcal{L}_a^s({}^nX; Y)$.

(c) $s : \mathcal{L}_a({}^nX; Y) \to \mathcal{L}_a^s({}^nX; Y) : L \mapsto s(L)$ is an idempotent mapping, i.e. $s^2 = s$.

(d) s is a linear operator.

(e) If $L \in \mathcal{L}_a({}^nX;Y)$ and $x \in X$, then $L(x^n) = s(L)(x^n)$.

It follows from (a), (b), (c) and (d) that $s : \mathcal{L}_a({}^nX;Y) \to \mathcal{L}_a^s({}^nX;Y)$ is a surjective linear projection. Also, if we let $\hat{L} = L \circ \Delta_n$, then it is clear from (e) that

$$\begin{aligned} \widehat{s(L)}(x) &= (s(L) \circ \Delta_n)(x) \\ &= s(L)(x^n) \\ &\overset{(e)}{=} L(x^n) \\ &= (L \circ \Delta_n)(x) \\ &= \hat{L}(x) \end{aligned}$$

for all $x \in X$.

We call $s(L)$ the symmetrization of L. Clearly, since $\widehat{s(L)} = \hat{L}$ for all $L \in \mathcal{L}_a({}^nX;Y)$, we have $P \in \mathcal{P}_a({}^nX;Y)$ if and only if there exists a symmetric n-linear mapping $L \in \mathcal{L}_a^s({}^nX;Y)$ such that $P = L \circ \Delta_n$, i.e. such that $P = \hat{L}$.
Given $L \in \mathcal{L}_a^s({}^nX;Y)$ we obtain a $P \in \mathcal{P}_a({}^nX;Y)$ by $P = \hat{L}$. Suppose $P \in \mathcal{P}_a({}^nX;Y)$ is given. How do we obtain $L \in \mathcal{L}_a^s({}^nX;Y)$ such that $P = \hat{L}$?

The answer is given by the following classical result:

Corollary 6.1.7 *(cf. [31], Corollary 1.6)(Polarization formula) Let $P \in \mathcal{P}_a({}^nX;Y)$, where X and Y are vector spaces over $\mathbb{C}$, and let $L \in \mathcal{L}_a^s({}^nX;Y)$. If $\hat{L} = P$, then*

$$L(x_1, x_2, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \epsilon_2 \ldots \epsilon_n P(\sum_{i=1}^{n} \epsilon_i x_i).$$

Notation 6.1.8 *(a) We let $\mathcal{P}({}^nX;Y)$, $\mathcal{L}({}^nX;Y)$ and $\mathcal{L}^s({}^nX;Y)$ denote respectively the spaces of continuous n-homogeneous polynomials from X into Y, the continuous n-linear mappings from X into Y and the continuous symmetric n-linear mappings from X into Y. If $Y = \mathbb{C}$, these spaces are denoted by $\mathcal{P}({}^nX)$, $\mathcal{L}({}^nX)$ and $\mathcal{L}^s({}^nX)$ respectively. In case of $n = 1$, we write $X^* = \mathcal{L}(X) = \mathcal{L}(X, \mathbb{C})$, i.e. X^* is the (continuous) dual space of X.*

(b) We denote the mapping

$$\mathcal{L}^s({}^nX;Y) \to \mathcal{P}({}^nX;Y): \quad L \mapsto \hat{L}$$

by $^\wedge$, *i.e.* $^\wedge(L) = \hat{L}$. *Since this is an isomorphism by Corollary 6.1.7, we may consider the inverse* $^\vee : \mathcal{P}({}^nX;Y) \to \mathcal{L}^s({}^nX;Y)$ *such that* $^\vee(P) = L$, *where* $\hat{L} = P$ *(or where* L *is given by the the Polarization formula (Corollary 6.1.7)). Hence* $\hat{L} = P$ *and* $\check{P} = L$.

Using the Binomial Theorem and the one variable Maximum Modulus Theorem (refer to the proof in [31] on page 12), the following can be derived:

Lemma 6.1.9 *(cf. [31], Lemma 1.9) Let* X *and* Y *be vector spaces over* $\mathbb{C}$, $P \in \mathcal{P}_a({}^nX;Y)$ *and* $x, y \in X$. *Then*

(a) $P(x+y) = \sum\limits_{j=0}^{n} \binom{n}{j} \check{P}(x^j, y^{n-j})$

$P(x) - P(y) = \sum\limits_{j=0}^{n-1} \binom{n}{j} \check{P}(y^j, (x-y)^{n-j})$.

(b) If $F = \mathbb{C}$, *then* $\sup\limits_{|\lambda| \leq 1} |P(x+\lambda y)|^2 \geq |P(x)|^2 + |P(y)|^2$.

Definition 6.1.10 *(cf. [31], page 14) If* B *is a convex balanced subset of the vector space* X, *we consider the Minkowski functional* $\| \cdot \|_B$ *of* B, *given by*

$$\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\} \text{ and}$$

$$\|x\|_B = \infty \text{ if } x \notin \lambda B \text{ for all } \lambda > 0 \text{ where } x \in X.$$

We have

$$\{x : \|x\|_B < 1\} \subset B \subset \{x : \|x\|_B \leq 1\}.$$

Consider the subspace $X_B = \bigcup\limits_{n>0} nB$ of X. Then $\| \cdot \|_B$ is a seminorm on X_B.

It is proved in [31] (Lemma 1.10, page 14) that:

Lemma 6.1.11 *Let* X *and* Y *be vector spaces over* $\mathbb{C}$, $P \in \mathcal{P}_a({}^nX;Y)$, $B \subset X$ *and let* α *denote a seminorm on* Y. *Then:*

(a) For a balanced $A \subset X$*, let*
$\|P\|_{\alpha,A} := \sup_{y \in A} \alpha(P(y))$ *and* $\|\check{P}\|_{\alpha,A^n} = \sup_{(y_1,\cdots,y_n)\in A^n} \alpha(\check{P}(y_1,\cdots,y_n))$.
If B *is balanced and* $x \in X$*, then* $\|P\|_{\alpha,B} \leq \|P\|_{\alpha,x+B}$*, where* $x + B = \{x + y : y \in B\}$.

(b) If B *is convex and balanced,* $\lambda \in \mathbb{R}$*,* $\lambda > 0$ *and* $\lambda x \in B$*, then*
$\|P\|_{\alpha,x+B} \leq (1 + \frac{1}{\lambda})^n \|P\|_{\alpha,B}$.

(c) If B *is convex and balanced,* x *and* y *in* B*,* $\|y\|_B \leq \varepsilon$ *and* $\|x-y\|_B \leq 1$*, then* $\alpha(P(x) - P(y)) \leq \frac{n^n}{n!}(1+\varepsilon)^n \|x-y\|_B \|P\|_{\alpha,B}$.

Using Lemma 6.1.11 it is proved in ([31], page 15) that

Proposition 6.1.12 *Let* X *be a locally convex space,* Y *a normed linear space and suppose* $P \in \mathcal{P}_a(^nX;Y)$*. The following are equivalent:*

(a) P *is locally uniformly continuous (i.e. for each* $x \in X$ *there exists a neighbourhood* V *of* x *such that* $P|_V$ *is uniformly continuous).*

(b) P *is everywhere continuous.*

(c) P *is continuous at some point.*

(d) P *is bounded on a neighbourhood of some point in* X*.*

(e) P *is a locally bounded function (i.e. every point in* X *contains a neighbourhood on which* P *is bounded).*

Corollary 6.1.13 *(cf. [31], Corollary 1.12) Let* X *and* Y *be locally convex spaces over* $\mathbb{C}$ *and let* $P \in \mathcal{P}_a(^nX;Y)$*. Then* $P \in \mathcal{P}(^nX;Y)$ *if and only if* P *is continuous at one point.*

Next we consider a useful factorisation lemma. If X and Y are locally convex spaces, α is a continuous seminorm on X and $P \in \mathcal{P}(^nX_\alpha;Y)$, then

$$\pi_\alpha^*(P) := P \circ \pi_\alpha \in \mathcal{P}(^nX;Y), \text{ i.e.}$$

$\pi_\alpha^* : \mathcal{P}(^nX_\alpha;Y) \to \mathcal{P}(^nX;Y) : P \mapsto P \circ \pi_\alpha$.
Denote by $cs(X)$ the set of all continuous seminorms on X. Of course the union

$$\bigcup_{\alpha \in cs(X)} \pi_\alpha^*(\mathcal{P}(^nX_\alpha;Y))$$

can (by the above identification) then be considered to be a subset of $\mathcal{P}(^nX;Y)$. Moreover, when Y is a normed linear space we have the following "Factorization Lemma"

Lemma 6.1.14 *(cf. [31], Lemma 1.13)(Factorization Lemma) If X is a locally convex space and Y is a normed space, then*

$$\mathcal{P}(^nX;Y) = \bigcup_{\alpha \in cs(X)} \pi_\alpha^*(\mathcal{P}(^nX_\alpha;Y)).$$

Some interesting results concerning polynomials on Banach spaces with the DPP are considered in ([31]). It is for instance proved in ([31]) that:

Proposition 6.1.15 *Let X be a Banach space with the Dunford-Pettis property and $P \in \mathcal{P}(^nX)$, then P is weakly sequentially continuous.*

Corollary 6.1.16 *If X is a Banach space with the Dunford-Pettis property and ℓ_1 is not isomorphic to a subspace of X, then continuous polynomials on X are (uniformly) weakly continuous on bounded sets.*

6.2 Holomorphic mappings

We now refer to [53] for the following information.

Definition 6.2.1 *Let X and Y be Banach spaces. A power series from X into Y around the point $a \in X$ is a series of mappings of the form $\sum_{m=0}^{\infty} P_m(x-a)$, where $P_m \in \mathcal{P}_a(^mX;Y)$ for every $m \in \mathbb{N} \cup \{0\}$ (i.e. the m-th term in the series is the value of a $m-1$-homogeneous polynomial at the point $x - a \in X$ and the series converges in the norm of Y). Since we know that $P_m = A_m \circ \Delta_m$ for some $A_m \in \mathcal{L}_a^s(^mX;Y)$ (unique), we have $\sum_{m=0}^{\infty} P_m(x-a) = \sum_{m=0}^{\infty} A_m((x-a)^m)$; hence $\hat{A}_m = P_m$ for all $m \in \mathbb{N} \cup \{0\}$.*

As usual, the radius of (uniform) convergence of the power series

$$\sum_{m=0}^{\infty} P_m(x-a)$$

is the supremum of all $r \geq 0$ such that the series converges uniformly on the ball $\bar{B}(a,r)$. As we would expect, the radius of convergence R is also given by the Cauchy-Hadamard Formula

$$\frac{1}{R} = \lim_{m \longrightarrow \infty} \sup \|P_m\|^{\frac{1}{m}}.$$

The reader is referred to ([53], Theorem 4.3) for the proof of this fact and that the series converges uniformly on $\bar{B}(a,r)$ whenever $0 \leq r < R$. If for some $0 < r < R$ we have $\sum_{m=0}^{\infty} P_m(x-a) = 0$ for all $x \in B(a,r)$, then $P_m = 0$ for every $m \in \mathbb{N} \cup \{0\}$ (cf. [53], Proposition 4.4).
For the discussion on holomorphic mappings to follow, we assume X and Y to be complex Banach spaces.

Definition 6.2.2 *Let U be an open subset of X. A mapping $f : U \to Y$ is said to be holomorphic (or analytic) if for each $a \in U$ there exists a ball $B(a,r) \subset U$ and a sequence of polynomials $P_m \in \mathcal{P}(^n X; Y)$ such that*

$$f(x) = \sum_{m=0}^{\infty} P_m(x-a),$$

uniformly on $B(a,r)$.

Notation 6.2.3 *The vector space of all holomorphic mappings from U into Y is denoted by $\mathcal{H}(U;Y)$. When $Y = \mathbb{C}$, we write $\mathcal{H}(U;\mathbb{C}) = \mathcal{H}(U)$.*

If $f(x) = \sum_{m=0}^{\infty} P_m(x-a) = \sum_{m=0}^{\infty} Q_m(x-a)$, for all $x \in B(a,r)$ as in Definition 6.2.2, then

$$\sum_{m=0}^{\infty} (P_m - Q_m)(x-a) = 0, \text{ with } P_m - Q_m \in \mathcal{P}(^m X; Y), \text{ i.e. } P_m = Q_m \text{ for all } m.$$

Therefore the sequence (P_n) in the power series expansion depends only on f and $a \in U$.
We call the unique power series expansion of f around a the Taylor series of f at a; sometimes the notation $P_m = P^m f(a)$ is used, i.e.

$$\sum_{m=0}^{\infty} P_m(x-a) = \sum_{m=0}^{\infty} P^m f(a)(x-a).$$

Moreover, the unique member $\check{P}_m \in \mathcal{L}_s({}^mX;Y)$ such that $P_m = \check{P}_m \circ \Delta_m$ is in this notation usually denoted by $A^m f(a)$, i.e.

$$f(x) = \sum_{m=0}^{\infty} A^m f(a)(x-a), \text{ with } (A^m f(a))^\wedge = P_m = P^m f(a).$$

Remark 6.2.4 *$\mathcal{P}(X;Y) \subset \mathcal{H}(X;Y)$, i.e. each polynomial (as a linear combination of m-homogeneous polynomials) is holomorphic.*

Remark 6.2.5 *Many properties of holomorphic mappings in Banach spaces can be derived from corresponding properties of holomorphic functions of one complex variable with the aid of the following result (cf. [53], Lemma 5.6): Let U be an open subset of X and let $f \in \mathcal{H}(U;Y)$. Then:*

(a) f is continuous.

(b) f is locally bounded, that is, f is bounded on a suitable neighbourhood of each point of U.

(c) For each $a \in U$, $b \in X$ and $\phi \in Y^*$, the function

$$\lambda \mapsto \phi \circ f(a + \lambda b)$$

is holomorphic on the open set $\{\lambda \in \mathbb{C} : a + \lambda b \in U\}$.

6.3 Dunford-Pettis-type functions

We recall from Chapter 2 that an operator $T : E \to X$ from a Banach lattice to a Banach space is said to be disjoint p-convergent if for every disjoint sequence $(x_n) \in \ell_p^{weak}(E)$, we have that $\|Tx_n\| \overset{n}{\underset{\infty}{\longrightarrow}} 0$.

We also recall the following notions:

Definition 6.3.1 *A subset A of X is said to be a Dunford-Pettis (respectively limited) set if for all $(x_n^*) \in c_0^{weak}(X^*)$ (respectively $(x_n^*) \in c_0^{weak^*}(X^*)$), we have that $\sup_{x \in A} |x_n^*(x)| \overset{n}{\underset{\infty}{\longrightarrow}} 0$. A sequence (x_n) is said to be Dunford-Pettis (respectively, limited) if the set $\{x_n : n \in \mathbb{N}\}$ is Dunford-Pettis (respectively, limited).*

Following [21] among others, we now introduce:

Definition 6.3.2 *A Banach lattice E is said to have the disjoint Dunford-Pettis property of order p (disjoint DPP_p for short) if every disjoint weakly p-summable sequence in E is Dunford-Pettis.*

We characterise the disjoint DPP_p as follows:

Theorem 6.3.3 *A Banach lattice E has the disjoint DPP_p if and only if $x_n^*(x_n) \xrightarrow[\infty]{n} 0$ for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ and all $(x_n^*) \in c_0^{weak}(E^*)$.*

Proof Suppose E has the disjoint DPP_p. Let $(x_n) \in \ell_p^{weak}(E)$ be a disjoint sequence. By the assumption the set $\{x_n : n \in \mathbb{N}\}$ is Dunford-Pettis. Thus, for $(x_n^*) \in c_0^{weak}(E^*)$ we have $\sup_k |x_n^*(x_k)| \xrightarrow[\infty]{n} 0$. In particular, $x_n^*(x_n) \xrightarrow[\infty]{n} 0$.

Conversely, assume $x_n^*(x_n) \xrightarrow[\infty]{n} 0$ for all $(x_n^*) \in c_0^{weak}(E^*)$ and all disjoint sequences $(x_n) \in \ell_p^{weak}(E)$. Suppose E does not have the disjoint DPP_p. Then there exists a disjoint $(x_n) \in \ell_p^{weak}(E)$ which is not Dunford-Pettis, i.e. there exists $(x_n^*) \in c_0^{weak}(E^*)$ such that

$$\sup_k |x_n^*(x_k)| \nrightarrow 0 \text{ if } n \to \infty.$$

Thus we may find a subsequence of (x_n^*), which we denote by (x_n^*) again, such that for all $n \in \mathbb{N}$,

$$\sup_k |x_n^*(x_k)| \geq \varepsilon. \tag{6.1}$$

Let $n_1 = 1$ and let $x_{k_1} \in \{x_n : n \in \mathbb{N}\}$ be such that

$$|x_n^*(x_{k_1})| \geq \frac{\varepsilon}{2}.$$

Since $x_n^*(x_j) \xrightarrow[\infty]{n} 0$ for all $1 \leq j \leq k_1$, we may choose an index $n_2 > n_1$ such that $|x_{n_2}^*(x_j)| < \dfrac{\varepsilon}{2}$ for all $j = 1, 2, \cdots k_1$.
By (6.1) there has to be $x_{k_2} \in \{x_n : n \in \mathbb{N}\}$ such that $|x_{n_2}^*(x_{k_2})| \geq \dfrac{\varepsilon}{2}$ and

where $k_2 > k_1$.
Simarly, since $x_n^*(x_j) \xrightarrow[\infty]{n} 0$ for all $1 \leq j \leq k_2$, we can find an index $n_3 > n_2 > n_1$ so that

$$|x_{n_3}^*(x_j)| < \frac{\varepsilon}{2} \text{ for } j = 1, 2, \cdots k_2.$$

Again by (6.1) there has to be $x_{k_3} \in \{x_n : n \in \mathbb{N}\}$ such that $|x_{n_3}^*(x_{k_3})| \geq \frac{\varepsilon}{2}$ and where $k_3 > k_2 > k_1$.
Continuing in this way, we may construct two "new" sequences $(x_n^*) \in c_0^{weak}(E^*)$ and $(x_n) \in \ell_p^{weak}(E)$ (where the x_n are mutually disjoint) such that

$$|x_n^*(x_n)| \geq \frac{\varepsilon}{2}.$$

This contradicts our assumption. □

We extend this notion by introducing the following stronger version:

Definition 6.3.4 *A Banach lattice E is said to have the disjoint Dunford-Pettis* property of order p (disjoint DP^*P_p for short) if every disjoint weakly p-summable sequence in E is limited.*

This definition leads to the following characterisation and the proof thereof (which we shall omit) is similar to that of Theorem 6.3.3:

Theorem 6.3.5 *A Banach lattice E has the disjoint DP^*P_p if and only if $x_n^*(x_n) \xrightarrow[\infty]{n} 0$ for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ and all $(x_n^*) \in c_0^{weak^*}(E^*)$.*

We agree to denote the family of disjoint p-convergent operators from a Banach lattice E to a Banach space X by $\mathcal{DC}_p(E, X)$. Following [37], we may also characterise the disjoint DP^*P_p as follows:

Proposition 6.3.6 *Let $1 \leq p < \infty$. A Banach lattice E has the disjoint DP^*P_p if and only if every operator $T \in \mathcal{L}(E, c_0)$ is disjoint p-convergent, i.e. if and only if $\mathcal{L}(E, c_0) = \mathcal{DC}_p(E, c_0)$.*

Proof Assume that $\mathcal{L}(E, c_0) = \mathcal{DC}_p(E, c_0)$. Given a disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ and a sequence $(x_n^*) \in c_0^{weak^*}(E^*)$, the operator $T : E \to c_0 : x \mapsto (x_n^*(x))$ is bounded; i.e. by assumption $\|Tx_n\| \xrightarrow[\infty]{n} 0$.

Therefore, $|x_n^*(x_n)| \leq \sup_j |x_j^*(x_n)| = \|Tx_n\| \overset{n}{\underset{\infty}{\longrightarrow}} 0$. By definition, E has the disjoint DP^*P_p.

Conversely, assume that E has the disjoint DP^*P_p and let $T \in \mathcal{L}(E, c_0)$. If $T \notin \mathcal{DC}_p(E, c_0)$, then for some disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ we may assume that there exists a $\delta > 0$ such that $\|Tx_m\| \geq \delta$ for all $m \in \mathbb{N}$. Denote by $\pi_j : c_0 \to \mathbb{K}$ the j^{th} coordinate projection. For each $m \in \mathbb{N}$, let $k_m \in \mathbb{N}$ (with $k_1 < k_2 < \cdots$) be such that $\frac{\delta}{2} < |\pi_{k_m}(Tx_m)| \leq \|Tx_m\|$. Put $x_i^* = \pi_i \circ T$. Then $Tx = (x_i^*(x))_i$ for all $x \in E$ and $(x_i^*) \in c_0^{weak^*}(E^*)$, from which $\frac{\delta}{2} < |x_{k_m}^*(x_m)|$ for all $m \in \mathbb{N}$ follows. Since for each fixed index k we have $x_k^*(x_m) \to 0$ as $m \to \infty$, it follows that there has to be infinitely many different positive integers k_m, such that $\frac{\delta}{2} < |x_{k_m}^*(x_m)|$ as m runs through $\mathbb{N}$. The sequence (x_n) being limited, the assumption implies that $\sup_n |x_{k_m}^*(x_n)| \to 0$ if $m \to \infty$; in particular, this implies that $x_{k_m}^*(x_m) \overset{m}{\underset{\infty}{\longrightarrow}} 0$, which leads to a contradiction. $\square$

Following the discussion in [37] in connection with so called p-convergent functions on Banach spaces (they are functions which map weakly-p-summable sequences onto norm convergent sequences), we now introduce the notion of disjoint p-convergent functions on a Banach lattice and consider some applications to the characterisation of Banach lattices with the disjoint DP^*P_p. Sometimes the proofs in this section carry over from the discussion in [37] (and [69]); however, in most cases, we will discuss the (adjusted to the current context) proofs for the sake of completeness.

Definition 6.3.7 *Let $1 \leq p < \infty$. A function f from a Banach lattice E into a Banach space X is said to be disjoint p-convergent if it maps disjoint weak p-convergent sequences onto norm convergent sequences.*

Proposition 6.3.8 *Let $1 \leq p < \infty$, E be a Banach lattice and X be a Banach space and assume that X contains an isomorphic copy of c_0. If every $T \in \mathcal{L}(E, X)$ is disjoint p-convergent, then E has the disjoint DP^*P_p. In this case, every polynomial $P \in \mathcal{P}(^nE, X)$ is a disjoint p-convergent function for all $n \in \mathbb{N}$.*

Proof Let $T : E \to c_0$ be bounded and $j : c_0 \to X$ the isomorphic embedding. Then $j \circ T \in \mathcal{L}(E, X)$ is disjoint p-convergent. Since

$\|Tx_n\| = \|j(Tx_n)\| \xrightarrow[\infty]{n} 0$ for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$, T is also disjoint p-convergent. By proposition 6.3.6, E has the disjoint DP^*P_p.

The result about polynomials follows by induction on the degree of P: Let $n = 1$ be the degree of the n-homogeneous polynomial P , then $P \in \mathcal{P}(^1E, X)$ is a bounded linear operator and hence disjoint p-convergent by assumption.

Assume every polynomial $\mathcal{P}(^nE, X)$ is disjoint p-convergent. Let $P \in \mathcal{P}(^{n+1}E, X)$ and let $\check{P}$ be the unique symmetric bounded $(n+1)$-linear mapping from E^{n+1} into X such that $P(x) = \check{P}(x, x, \cdots x)$. Fix $x \in E$ and consider the unique $Q_x \in \mathcal{P}(^nE, X)$ such that $Q_x(z) = \check{P}(x, z, \cdots z) = \check{P}(x, z^n)$ for all $z \in E$. By the induction assumption, Q_x is disjoint p-convergent. Hence, if (x_m) is a disjoint weakly p-summable sequence, then $\lim_m \check{P}(x, x_m^n) = \lim_m Q_x(x_m) = 0$ in norm. Choose $\phi \in X^*$ such that $\|\phi_m\| = 1$ and $\phi_m(P(x_m)) = \|P(x_m)\|$. Then $T : E \to c_0 \subseteq X$, given by

$$T(x) = (\phi_m(\check{P}(x, x_m^n)))_m$$

is a well defined operator which is disjoint p-convergent since E has the disjoint DP^*P_p. Therefore,

$$0 = \lim_m \|T(x_m)\|_{c_0} = \lim_m \|(\phi_m(\check{P}(x_m, x_m^n)))_m\| = \lim_m \|P(x_m)\|_X,$$

as required.

Now, let (x_n) be a disjoint weakly p-convergent sequence of E at a point a and consider the identity

$$P(x_m) - P(a) = P(x_m - a) + \sum_{j=1}^{n} \binom{n+1}{j} \check{P}((x_m - a)^{n+1-j}, a^j).$$

By the above, $\lim P(x_m - a) = 0$ and moreover every mapping $z \in E \mapsto \check{P}(z^{n+1-j}, a^j) \in X$, $j = 1, \cdots n$, is a polynomial of degree not greater than n, so by induction $\lim_m \check{P}((x_m - a)^{n+1-j}, a^j) = 0$. Thus we conclude that $\lim_m P(x_m) = P(a)$. □

It is well-known (cf. [3], Theorem 4.34) that if W is a weakly relatively compact subset of a Banach lattice, then every disjoint sequence in the solid hull of W converges weakly to zero. In particular, this implies that all disjoint weak p-convergent sequences in a Banach lattice E have limit 0 and so they belong to $\ell_p^{weak}(E)$.

Definition 6.3.9 *We call a subset A of a Banach lattice E "disjoint weakly p-compact" if it is weakly p-compact and its elements are mutually disjoint.*

Proposition 6.3.10 *A Banach lattice E has the disjoint DP^*P_p if and only if all disjoint weakly p-compact sets in E are limited.*

Proof If all disjoint weakly p-compact sets in E are limited, then for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ the set $\{x_n : n \in \mathbb{N}\}$ is disjoint weakly p-compact and so it is limited. Therefore, for each $(x_n^*) \in c_0^{weak^*}(E^*)$ we have

$$|x_n^*(x_n)| \leq \sup_k |x_n^*(x_k)| \to 0 \text{ if } n \to \infty.$$

This proves that E has the disjoint DP^*P_p.

Conversely, suppose E has the disjoint DP^*P_p. Let $A \subset E$ be disjoint weakly p-compact. If we assume that A is not limited, then there exists a sequence $(x_n^*) \in c_0^{weak^*}(E^*)$ such that

$$\sup_{x \in A} |x_n^*(x)| \nrightarrow 0 \text{ if } n \to \infty.$$

Taking subsequences if necessary, we may assume that there exists $\varepsilon > 0$ and a sequence $(x_n) \subset A$ so that

$$(\dagger) \qquad\qquad |x_n^*(x_n)| \geq \varepsilon, \quad \forall n \in \mathbb{N}.$$

Since A is weakly p-compact, there is a weakly p-convergent subsequence $(x_{n_k})_k$ of (x_n). Since the sequence $(x_{n_k})_k$ consists of mutually disjoint elements, it follows that $(x_{n_k})_k \in \ell_p^{weak}(E)$. Since E has the disjoint DP^*P_p, we have

$$x_{n_k}^*(x_{n_k}) \xrightarrow[\infty]{n} 0,$$

contradicting (†) above. □

From Theorem 5 of the paper [40] we know that if a sequence of k-homogeneous polynomials $(P_n) \subset \mathcal{P}(^kX)$ converges pointwise to a limit P (also a k-homogeneous polynomial), then $P_n \xrightarrow[\infty]{n} P$ uniformly on all limited subsets of X. Using this fact, we have the following result in connection with pointwise convergence of k-homogeneous polynomials on Banach lattices with the disjoint DP^*P_p:

Corollary 6.3.11 *Let E be a Banach lattice with the disjoint DP^*P_p and let $(P_n) \subset \mathcal{P}(^kE)$ such that $P_n \xrightarrow[\infty]{n} P \in \mathcal{P}(^kE)$ pointwise. Then $P_n \xrightarrow[\infty]{n} P$ uniformly on all disjoint weakly p-compact sets in E and for each disjoint sequence $(x_n) \in \ell_p^{weak}(E)$, we have $P_n(x_n) \xrightarrow[\infty]{n} 0$.*

Proof Let $(P_n) \subset \mathcal{P}(^kE)$ such that $P_n \xrightarrow[\infty]{n} P \in \mathcal{P}(^kE)$ pointwise. Since E has the disjoint DP^*P_p, it follows that all disjoint weakly p-compact sets in E are limited. Therefore, by the result in [40] mentioned above, $P_n \xrightarrow[\infty]{n} P$ uniformly on all disjoint weakly p-compact sets in E. Let $(x_n) \in \ell_p^{weak}(E)$ be a disjoint sequence. By the assumption that E has the disjoint DP^*P_p, the sequence (x_n) is limited, i.e the set $A := \{x_n : n \in \mathbb{N}\}$ is limited. Therefore, $P_n \xrightarrow[\infty]{n} P$ uniformly on A. Let $\varepsilon > 0$ be given. There exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|P_n(y) - P(y)| < \frac{\varepsilon}{2}$ for all $n \geq n_0$ and for all $y \in A$; in particular, this says that $\sup_j |P_n(x_j) - P(x_j)| < \frac{\varepsilon}{2}$ for all $n > n_0$. Thus, we have $|P_n(x_n) - P(x_n)| < \frac{\varepsilon}{2}$ for all $n \geq n_0$. Also, since the disjoint sequence $(x_n) \in \ell_p^{weak}(E)$ is a limited set, it follows from Theorem 3 in [40] that P is weakly continuous on the sequence (x_n), i.e. $P(x_n) \xrightarrow[\infty]{n} 0$. We may thus assume that $|P(x_n)| < \frac{\varepsilon}{2}$ for all $n \geq n_0$. It follows that

$$|P_n(x_n)| \leq |P_n(x_n) - P(x_n)| + |P(x_n)| < \varepsilon$$

for all $n \geq n_0$. $\square$

We denote the class of all disjoint p-convergent k-homogeneous polynomials from E into X by $\mathbb{DP}_{pc}(^kE, X)$.

Proposition 6.3.12 *Let E be a Banach lattice. Then following assertions are equivalent:*

(1) E has the disjoint DP^*P_p.

(2) Every operator $T : E \to c_0$ is disjoint p-convergent.

(3) For all integers k, each polynomial $P \in \mathcal{P}(^kE, c_0)$ is disjoint p-convergent.

(4) For some integer k, each polynomial $P \in \mathcal{P}(^kE, c_0)$ is disjoint p-convergent.

Proof (1) $\Leftrightarrow$ (2) follows from Proposition 6.3.6.
(1) $\Rightarrow$ (3) follows from Proposition 6.3.6 and Proposition 6.3.8.
(3) $\Rightarrow$ (4) is trivial.
We prove (4) $\Rightarrow$ (2): Let $T \in \mathcal{L}(E, c_0)$ and define $\omega : E^k \to c_0$ by

$$\omega(x_1, x_2, \cdots x_k) = (Tx_1)(Tx_2)\cdots(Tx_k) = (\eta_{i,1}\eta_{i,2}\cdots\eta_{i,k})_{i\in\mathbb{N}},$$

where $Tx_j = (\eta_{i,j})_i$ for $j = 1, 2, \cdots k$. Clearly, $\omega \in \mathcal{L}({}^kE, c_0)$.

Thus if $\breve{\omega} = \omega \circ \Delta_k$, then $\breve{\omega} \in \mathcal{P}({}^kE, c_0)$.

By assumption, $\breve{\omega} \in \mathbb{DP}_{pc}({}^kE, c_0)$. Given a disjoint $(x_n) \in \ell_p^{weak}(E)$, $\lim_n \breve{\omega}(x_n) = 0$. This implies that $(Tx_n)^k \underset{\infty}{\overset{n}{\longrightarrow}} 0$ and hence that $\|Tx_n\|_{c_0} \underset{\infty}{\overset{n}{\longrightarrow}} 0$. □

The following Lemmas were discussed in [69]:

Lemma 6.3.13 *(cf. [69], Lemma 2.1.10) Let (X, d) be a metric space and denote the metric topology on X by τ_d. Let $A \subset X$ be a τ_d-compact subset of X. For any weaker topology τ on X such that (X, τ) is a Hausdorff topological space and any τ-convergent sequence (x_n) in A (i.e. $x_n \overset{\tau}{\to} x \in A$) we have $x_n \overset{\tau_d}{\to} x$.*

Lemma 6.3.14 *Let $f \in \mathcal{H}(X, Y)$ be bounded on weakly compact (resp. limited) sets in X. Then f may be approximated uniformly on weakly compact (resp. limited) sets in X by its Taylor series at 0.*

Lemma 6.3.15 *(cf. [69], Lemma 4.2.2) Let $f_n, f : X \to Y$ (with $n = 1, 2, \cdots$) be functions from X into Y. Suppose L is a subset of X such that each f_n is weakly continuous on L and $f_n \underset{\infty}{\overset{n}{\longrightarrow}} f$ uniformly on L. Then f is weakly continuous on L.*

Recall that in a Gelfand-Phillips space limited sets are relatively (norm) compact. For polynomials and holomorphic functions on Banach lattices with values in a Banach space with the Gelfand-Phillips property we have the following result:

Proposition 6.3.16 *If E has the disjoint DP^*P_p and X is a Gelfand-Phillips space, then every $P \in \mathcal{P}({}^nE, X)$ is disjoint p-convergent. Furthermore, each $f \in \mathcal{H}(E, X)$ which is bounded on limited sets, is weakly continuous on disjoint weakly p-compact sets.*

Proof Let $P \in \mathcal{P}(^nE, X)$. To show that P is disjoint p-convergent, assume that (x_n) is weakly p-summable and disjoint in E. Then the set $A := \{x_n : n \in \mathbb{N}\}$ is disjoint weakly p-compact, thus limited in E; hence, since P maps limited sets onto limited sets and X is a Gelfand-Phillips space, $\overline{P(A)}$ is norm compact in X. We show that $P(x_n) \xrightarrow[\infty]{n} 0$ weakly in X: Fore each $\phi \in X^*$, the function $\phi \circ P : X \to \mathbb{K}$ is a scalar-valued polynomial and so it is weakly continuous on limited sets. Therefore, $(\phi \circ P)(x_n) \xrightarrow[\infty]{n} 0$. Thus, $P(x_n) \xrightarrow[\infty]{n} 0$ weakly in X. By Lemma 6.3.13, $P(x_n) \xrightarrow[\infty]{n} 0$ in the norm of X. Thus it is clear that P is a disjoint p-convergent function.

The above argument actually also yields that each $P \in \mathcal{P}(^nE, X)$ is weakly sequentially continuous on limited sets. Therefore, by Theorem 7 in [40], P is weakly continuous on all limited sets, implying that it is also weakly continuous on all disjoint weakly p-compact sets in the space E (which has the disjoint DP^*P_p). Thus, it follows from Lemmas 6.3.14 and 6.3.15 that each $f \in \mathcal{H}(E, X)$ which is bounded on limited sets, is weakly continuous on disjoint weakly p-compact sets. □

Let E, F be Banach lattices. Recall that the ordering on $E \times F$ is defined by

$$(x_1, y_1) \leq (x_2, y_2) \iff x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

In this ordering we have:

(1) $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$;

(2) $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$;

(3) $|(x_1, y_1)| = (|x_1|, |x_2|)$.

Using the norm $\|(x, y)\| = \|x\| + \|y\|$ on the product space $E \times F$, then $(E \times F, \|(\ , \ )\|)$ defines a Banach lattice.

The following general result will be needed in our proof of yet another characterisation of Banach lattices with disjoint DP^*P_p to follow.

Lemma 6.3.17 *Let E, F be Banach lattices and $(x_n) \in E^{\mathbb{N}}$, $(y_n) \in F^{\mathbb{N}}$ Then:*

(i) (x_n, y_n) is a disjoint sequence in $E \times F$ if and only if (x_n) is a disjoint sequence in E and (y_n) is a disjoint sequence in F.

(ii) $(x_n, y_n) \in \ell_p^{weak}(E \times F)$ *if and only if* $(x_n) \in \ell_p^{weak}(E)$ *and* $(y_n) \in \ell_p^{weak}(F)$.

Proof The result (i) follows from

$$(0,0) = |(x_n, y_n)| \wedge |(x_m, y_m)| = (|x_n| \wedge |x_m|, |y_n| \wedge |y_m|), \quad \forall m \neq n.$$

We prove (ii): Let $\pi_E : E \times F \to E : (x, y) \mapsto x$ and $\pi_F : E \times F \to F : (x, y) \mapsto y$ be the bounded linear coordinate projections. For $x^* \in E^*$ we have $x^* \circ \pi_E \in (E \times F)^*$ and thus we have

$$\begin{aligned} (\langle x_n, x^* \rangle) &= (\langle \pi_E(x_n, y_n), x^* \rangle)_n \\ &= (\langle (x_n, y_n), x^* \circ \pi_E \rangle)_n \in \ell_p. \end{aligned}$$

This shows that $(x_n) \in \ell_p^{weak}(E)$. Similarly, $(y_n) \in \ell_p^{weak}(F)$.

Conversely, let $(x_n) \in \ell_p^{weak}(E)$ and $(y_n) \in \ell_p^{weak}(F)$. We show that $(x_n, y_n) \in \ell_p^{weak}(E \times F)$: Let $j_1 : E \to E \times F$ and $j_2 : F \to E \times F$ be the embedding mappings, i.e. $j_1(x) = (x, 0)$ and $j_2(y) = (0, y)$. They are bounded linear operators. Consider any $\phi \in (E \times F)^*$, then $\phi \circ j_1 \in E^*$ and $\phi \circ j_2 \in F^*$. Also,

$$\begin{aligned} \phi(x_n, y_n) &= \phi((x_n, 0) + (0, y_n)) \\ &= \phi(j_1(x_n) + j_2(y_n)) \\ &= (\phi \circ j_1)(x_n) + (\phi \circ j_2)(y_n) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Thus

$$\begin{aligned} (\phi(x_n, y_n))_{n \in \mathbb{N}} &= ((\phi \circ j_1)(x_n))_n + ((\phi \circ j_2)(y_n))_n \\ &\in \ell_p. \end{aligned}$$

This shows that $(x_n, y_n)_n \in \ell_p^{weak}(E \times F)$. □

Recall the following definition of separately compact bilinear operator:

Definition 6.3.18 *Let X, Y and Z be Banach spaces. A bilinear operator*

$$\phi : X \times Y \to Z$$

is called separately compact if for each fixed $y \in Y$, the linear operator $T_y : X \to Z : \ x \mapsto \phi(x, y)$ is compact and for each fixed $x \in X$, the linear operator $T_x : Y \to Z : \ y \mapsto \phi(x, y)$ is compact.

Lemma 6.3.19 *If $T \in \mathcal{L}(X, c_0)$ and $T \otimes T : X \times X \to c_0$ is given by*

$$(T \otimes T)(x, y) = T(x)T(y),$$

the coordinate-wise product of two sequences in c_0, then $T \otimes T$ is a separately compact bilinear operator.

Proof If $T \in \mathcal{L}(X, c_0)$ is given, then the mapping $T \otimes T$ is clearly bilinear and symmetric, i.e. $(T \otimes T)(x, y) = (T \otimes T)(y, x)$. We only need to prove that $T \otimes T$ is separately compact: For each fixed $x \in X$, the operator

$$T_x : X \to c_0 : \ y \mapsto T(x)T(y) = (T \otimes T)(x, y)$$

is linear and bounded with $\|T_x\| \leq \|T\|^2 \|x\|$. It may be assumed without loss of generality that $\|T\| = 1$. Since $T(x)$ is a sequence in c_0, we agree to denote the m-th term of $T(x)$ by $T(x)_m$. Then, for $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|T(x)_m| \leq \varepsilon$ for all $m \geq n$.
Denote $(T(x)_1, T(x)_2, \cdots, T(x)_n, 0, 0, \cdots)$ by $T(x)(\leq n)$ and $(0, 0, \cdots, 0, T(x)_{n+1}, T(x)_{n+2}, \cdots)$ by $T(x)(> n)$. Then

$$\begin{aligned} T_x(y) &= T(x)T(y) \\ &= T(x)(\leq n)T(y) + T(x)(> n)T(y). \end{aligned}$$

The set $\{T(x)(\leq n)T(y) : y \in X\}$ is contained in a finite-dimensional subspace

$$M := \{(\xi_i)(\leq n) : (\xi_i) \in c_0\}$$

of c_0. We have

$$\{T(x)(\leq n)T(y) : y \in B_X\} \subseteq \|T(x)\| B_M,$$

which is compact.

For each $\varepsilon > 0$ there exists $z_1, \cdots, z_p \in \|T(x)\| B_M$ such that

$$\|T(x)\| B_M \subseteq \bigcup_{i=1}^{p} (z_i + \varepsilon B_{c_0}).$$

Thus,

$$\begin{aligned} T_x(B_X) &= T(x)T(B_X) \\ &\subseteq \bigcup_{i=1}^{p}(z_i+\varepsilon B_{c_0})+\varepsilon B_{c_0} \\ &\subseteq \bigcup_{i=1}^{p+1}(z_i+2\varepsilon B_{c_0}) \text{ with } z_{p+1}=0. \end{aligned}$$

This shows that $T_x(B_X)$ is relatively compact, i.e. that T_x is compact. This implies that $T \otimes T$ is separately compact. □

Proposition 6.3.20 *Let E be a Banach lattice. If every symmetric bilinear separately compact map $E \times E \to c_0$ is disjoint p-convergent, then E has the disjoint DP^*P_p.*

Proof Consider any $T \in \mathcal{L}(E, c_0)$. Let $T \otimes T : X \times X \to c_0$ be the separately compact bilinear operator as in Lemma 6.3.19 above. By our assumption, $T \otimes T$ is therefore also disjoint p-convergent. Consider any disjoint sequence $(x_i) \in \ell_p^{weak}(E)$. We prove that

$$\|Tx_i\| \longrightarrow 0, \text{ if } i \longrightarrow \infty :$$

By Lemma 6.3.17, $(x_n, x_n)_n \in \ell_p^{weak}(E \times E)$ and (x_n, x_n) is a disjoint sequence. Since $T \otimes T$ is p-convergent, it follows that

$$\begin{aligned} \|Tx_n\|_{c_0}^2 &= \|(T\otimes T)(x_n,x_n)\|_{c_0} \\ &\xrightarrow[\infty]{n} \|(T\otimes T)(0,0)\|_{c_0} \\ &= 0. \end{aligned}$$

By Proposition 6.3.6, E has the disjoint DP^*P_p. □

We do not know whether the converse of the result in Proposition 6.3.20 is true.

Lemma 6.3.21 *Let E be a Banach lattice and X be a Banach space. If $f \in \mathcal{H}(E, X)$ is disjoint p-convergent, then it is bounded on all disjoint weakly p-compact subsets of E.*

Proof Suppose $f \in \mathcal{H}(E,X)$ is unbounded on a disjoint weakly p-compact subset A of X. Then for each $n \in \mathbb{N}$ there exists $x_n \in A$ such that $\|f(x_n)\| \geq n$. A being disjoint weakly p-compact, there exists a subsequence (x_{n_k}) such that $(x_{n_k}) \in \ell_p^{weak}(E)$. By assumption, $f(x_{n_k}) \longrightarrow 0$ if $k \to \infty$ in norm; this however, contradicts the fact that $\|f(x_{n_k})\| \geq n_k$ for all k. $\square$

Recall that a subset L of X is called bounding if every $f \in \mathcal{H}(X)$ is bounded on L. We denote by $\mathbb{B}_p$, the class of all Banach lattices whose disjoint weakly p-compact subsets are bounding.

Proposition 6.3.22 *Let E be a Banach lattice. If each $f \in \mathcal{H}(E)$ is disjoint p-convergent, then $E \in \mathbb{B}_p$. On the other hand, if $E \in \mathbb{B}_p$, then each $f \in \mathcal{H}(E)$ is weakly continuous on disjoint weakly p-compact sets.*

Proof Assume each $f \in \mathcal{H}(E)$ is disjoint p-convergent. Then by Lemma 6.3.21 each $f \in \mathcal{H}(E)$ is bounded on all disjoint weakly p-compact subsets of E, i.e. all disjoint weakly p-compact subsets of E are bounding. Thus $E \in \mathbb{B}_p$.

Conversely, let $E \in \mathbb{B}_p$. Then all the disjoint weakly p-compact subsets of E are bounding, hence they are limited. Thus E has the disjoint DP^*P_p. Since $\mathbb{K}$ is a Gelfand-Phillips space, it follows from Proposition 6.3.16 that each $f \in \mathcal{H}(E)$ is weakly continuous on disjoint weakly p-compact sets. $\square$

Proposition 6.3.23 *Let E be a Banach lattice and X be a Banach space, and let $E \in \mathbb{B}_p$ and X be a Gelfand-Phillips space. Then each $f \in \mathcal{H}(E,X)$ is disjoint p-convergent.*

Proof As in the proof of Proposition 6.3.22, we see that E has the disjoint DP^*P_p. Also, since $E \in \mathbb{B}_p$ it follows that each $f \in \mathcal{H}(E,X)$ is bounded on the disjoint weakly p-compact subsets of E: for each $f \in \mathcal{H}(E,X)$ and each $y^* \in X^*$ we have $y^* \circ f \in \mathcal{H}(E)$, showing that for each disjoint weakly p-compact subset of A of E, the set $f(A)$ is weakly bounded and hence bounded in X. Disjoint weakly p-compact subsets of E are limited (since E has the disjoint DP^*P_p) and therefore it follows from Lemma 6.3.14 that f can be approximated uniformly on all disjoint weakly p-compact subsets of E by its Taylor series at 0, i.e. $f_n \overset{n}{\underset{\infty}{\longrightarrow}} f$ uniformly on all disjoint weakly p-compact subsets of E, where each $f_n(x) = \sum_{k=0}^{n} P_k(x)$ is the n-th partial sum of the Taylor series expansion $f(x) = \sum_{k=0}^{\infty} P_k(x)$ of f. We show that

f is disjoint p-convergent:
Let $(x_n) \subset E$ be a disjoint and weakly p-convergent sequence, i.e. $(x_n)_n \in \ell_p^{weak}(E)$. Now $f_n \overset{n}{\underset{\infty}{\longrightarrow}} f$ uniformly on the disjoint weakly p-compact set $L = \{x_n : n \in \mathbb{N}\}$. So, for given $\varepsilon > 0$ we let $n_0 \in \mathbb{N}$ such that

$$\|f_n(x) - f(x)\| < \frac{\varepsilon}{2}, \quad \forall n \geq n_0, \forall x \in L.$$

Each P_k being disjoint p-convergent by Proposition 6.3.16, it follows that each f_n (hence also f_{n_0}) is disjoint p-convergent. Therefore, $f_{n_0}(x_n) \longrightarrow 0$ in norm if $n \longrightarrow \infty$, i.e. there exists $k_0 \in \mathbb{N}$ such that $\|f_{n_0}(x_n)\| < \frac{\varepsilon}{2}$ for all $n \geq k_0$. It follows for all $n \geq k_0$ that

$$\|f(x_n))\| \leq \|f(x_n) - f_{n_0}(x_n)\| + \|f_{n_0}(x_n)\| < \varepsilon.$$

Thus we have verified that each $f \in \mathcal{H}(E, X)$ is disjoint p-convergent. $\square$

Since $\mathbb{K}$ is a Gelfand-Philips space, we conclude from Proposition 6.3.22 and Proposition 6.3.23 that

Corollary 6.3.24 *Let E be a Banach lattice. Then each $f \in \mathcal{H}(E)$ is disjoint p-convergent, if and only if $E \in \mathbb{B}_p$.*

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