Invariant Solutions and Conservation Laws for Soil Water Redistribution and Extraction Flow Models

by

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Declaration

I declare that the dissertation for the degree of Master of Science at the University of North-West hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

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Abstract

In this dissertation we use Lie symmetry analysis to obtain invariant solutions for certain soil water equations. These solutions are invariant under two-parameter symmetry groups obtained by the group classification of the governing equation. We also obtain all nontrivial conservation laws for a class of (2+1) nonlinear evolution partial differential equations which are related to the soil water equations. It is shown that nontrivial conservation laws exist for certain classes of equations which admit point symmetries. We note that one cannot invoke Noether's theorem here as there is no Lagrangian for these partial differential equations.

Introduction

A class of non-linear partial differential equations which models soil water infiltration and redistribution in a bedded soil profile irrigated by a drip irrigation system is described by

$$C(\psi)\psi_t = (K(\psi)\psi_x)_x + (K(\psi)(\psi_z - 1))_z - S(\psi) \qquad (1)$$

where $C(\psi) \neq 0$, $K(\psi) \neq 0$ and $S(\psi)$ are three arbitrary functions. Here ψ is a soil moisture pressure head, $C(\psi)$ is a specific water capacity, $K(\psi)$ is an unsaturated hydraulic conductivity, $S(\psi)$ is a sink or source term, t is a time, x is a horizontal and z is vertical axis, which is considered positive downward (see [1] and [2]).

Much work has been done on infiltration from line sources (see eg. [2] and references there in), but most of it is limited to the solution of the linearized, steady-state form of the flow equation. Analytical and numerical solutions of equation (1) for the functions $C(\psi)$ and $K(\psi)$ as constants and $S(\psi)$ as a linear function are given in the literature.

Using group theoretic approach, Baikov *et al.* [3] (see also [4], Vol. 2, Chapter 2) studied equation (1) for special coefficients $C(\psi)$, $K(\psi)$ and $S(\psi)$ which are not constants nor linear. Lie group classification of equation (1) with respect to admitted point transformation groups was done in [3] and invariant solutions for two particular equations of form (1) were also presented.

Exact/asymptotic invariant solutions of equation (1) for some particular types of the coefficients $C(\psi)$, $K(\psi)$ and $S(\psi)$ when an extension of the principal Lie algebra L_p occurs have also been obtained by Baikov and Khalique [5].

Conservation laws for some classes of soil water equations were obtained and their association with the generators of Lie symmetries were given in Kara and Khalique [6].

In this dissertation we obtain invariant solutions for certain soil water equations (1). These solutions are invariant under two-parameter symmetry groups obtained by the group classification of the governing equation. We also obtain all nontrivial conservation laws for a class of (2+1) nonlinear evolution partial differential equations which are related to the soil water equations (1).

In more detail, the outline of the research project is as follows:

In chapter 1, we recall the basic definitions and theorems on the one-parameter groups of transformations and present the notation that we will use in this project. We also outline a few results on symmetry groups and conservation laws.

In chapter 2, we find invariant solutions of some classes of soil water equations.

In chapter 3, using the direct method, we determine all nontrivial conserved vectors for a class of (2+1) nonlinear evolution partial differential equations which contain three arbitrary elements and are related to the soil water equations.

Chapter 1

Symmetries of differential equation

1.1 Introduction

Formulation of fundamental natural laws and technological problems are prevalent in terms of differential equations. These equations relate the behaviour of certain unknown functions at a given point to their behaviour at neighbouring points. In order to solve these equations we use symmetry and theory of continuous groups.

The idea of symmetry spreads through every part of the mathematical models, especially those ones formulated in terms of differential equations. Application of theory of continuous groups, which combines algebra, analysis and geometry together with symmetry reveals some knowledge of facts which had been unknown about differential equations.

This theory (theory of continuous groups) was originated and elaborated by an outstanding mathematician of the nineteenth century, Sophus Lie (1844-1899). It was the final outcome of about three centuries of efforts by renowned mathematicians to solve algebraic equations by radicals.

Lie group analysis based on symmetry and invariance principles is the only systematic method for solving nonlinear differential equations analytically [11].

In this chapter, we present some of the most important parts of Lie theory of transformation groups as applied to differential equations. There are several books on this area, e.g., Ovsiannikov [7], Olver [8], Bluman and Kumei [9], Stephani [10], Ibragimov [11] and the original source Lie (see, e.g., one of Lie's several contributions Lie [12]), or more recently, Cantwell [13] and Mahomed [14]. One can also refer to the CRC Handbooks on Lie group analysis of differential equations, edited by Ibragimov [4].

1.2 Notations, definitions and main result

In this section we give the notation as used by Mahomed [14]. Consider a kth-order ($k \geq 1$) system of differential equations

$$E^{\sigma}(x, u, u_{(1)}, ..., u_{(k)}) = 0, \quad \sigma = 1, ..., \bar{m}, \tag{1.1}$$

where $u = (u^1, ..., u^m)$, the dependent variable, is a function of the independent variable $x = (x^1, ..., x^n)$ and $u_{(1)}, u_{(2)}$ up to $u_{(k)}$ are the collection of all first, second up to kth-order partial derivatives: $u_{(1)} = \{u_i^{\alpha}\}$, $u_{(2)} = \{u_{ij}^{\alpha}\}, ..., u_{(k)} = \{u^{\alpha}_{i_1 ... i_k}\}$ for $\alpha = 1, ..., m; i, j, i_1 ..., i_k = 1, ..., n$. We also assume that the rank of the Jacobian matrix $\left(\frac{\partial E^v}{\partial x^i}, ..., \frac{\partial E^v}{\partial u^{\alpha}_{i_1...i_k}}\right)$ is $\bar{m}$ on (1.1). The maximal order of the equations appearing in (1.1) is k. Usually in applications $\bar{m} = m$. If x is a single independent variable, then (1.1) becomes a system of ordinary differential equations, otherwise it is a system of partial differential equations.

Definition 1 We say that a transformation of the variables x and u, namely,

$$\bar{x}^i = f^i(x, u), \quad \bar{u}^{\alpha} = \phi^{\alpha}(x, u), \quad i = 1, ..., n; \quad \alpha = 1, ..., m, \tag{1.2}$$

is a symmetry transformation of the system (1.1) if (1.1) is form-invariant in the new variables $\bar{x}$ and $\bar{u}$.

Definition 2 We say that a set G of transformations

$$T_a : \bar{x}^i = f^i(x, u, a), \quad \bar{u}^{\alpha} = \phi^{\alpha}(x, u, a), \quad i = 1, ..., n, \quad \alpha = 1, ..., m, \tag{1.3}$$

where a is a real parameter which continuously ranges in values from a neighbourhood $D \subset R$ of $a = 0$ and f^i, ϕ^α are differentiable functions, is a continuous one-parameter (local) Lie group of transformations in R^{n+m} if the following properties are satisfied:

(i) Identity: $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$.

(ii) Closure: For T_a, T_b in $G, a, b \in D' \subset D$,

$$T_b T_a = T_c \in G, \quad c = \varphi(a, b) \in D. \tag{1.4}$$

(iii) Inverses: For $T_a \in G, a \in D' \subset D, T_a^{-1} = T_{a^{-1}} \in G, a^{-1} \in D$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$.

From (ii) the associativity property follows. Also, if the identity transformation occurs at $a = a_0 \neq 0$, i.e., T_{a_0} is the identity, then a shift of the parameter $a = \bar{a} + a_0$ will give T_0 as above. One can write the group property (ii) as

$$f^i(\bar{x}, \bar{u}, b) = f^i(x, u, \varphi(a, b)), \quad \phi^\alpha(\bar{x}, \bar{u}, b) = \phi^\alpha(x, u, \varphi(a, b)). \tag{1.5}$$

We say that a group parameter a is canonical if $\varphi(a, b) = a + b$ and we have the following theorem.

Theorem 1 For any $\varphi(a, b)$, there exist the canonical parameter

$$\bar{a} = \int_0^a \frac{da'}{A(a')},$$

where

$$A(a) = \frac{\partial \varphi(a, b)}{\partial b}|_{b=0}.$$

From now on, we consider one-parameter groups with the canonical parameter. If the transformations (1.3) of a group G are symmetry transformations of the equation (1.1), then G is called a symmetry group of (1.1). According to Lie's theory, the construction of a one parameter group G is equivalent to the determination of the corresponding infinitesimal transformations:

$$\bar{x}^i \approx x^i + a\xi^i(x, u), \quad \bar{u}^\alpha \approx u^\alpha + a\eta^\alpha(x, u). \tag{1.6}$$

These are obtained by the Taylor series expansion in a of (1.3) about $a = 0$ taking into account the initial conditions $f^i|_{a=0} = x^i$, $\phi^\alpha|_{a=0} = u^\alpha$. Thus

$$\xi^i(x,u) = \frac{\partial f^i(x,u,a)}{\partial a}|_{a=0}, \quad \eta^\alpha(x,u) = \frac{\partial \phi^\alpha(x,u,a)}{\partial a}|_{a=0}. \tag{1.7}$$

We introduce the symbol X of the infinitesimal transformations by writing (1.6) as

$$\bar{x}^i \approx (1 + aX)x^i, \quad \bar{u}^\alpha \approx (1 + aX)u^\alpha,$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} . \tag{1.8}$$

X is also known as the infinitesimal operator or generator of the group G. If G is admitted by (1.1), then X is an admitted operator of (1.1). By means of the following Lie's theorem, one-parameter groups can be obtained from their generators:

Theorem 2 Given the infinitesimal transformations (1.6) or its symbol X, the corresponding one-parameter group G is obtained by solution of the Lie equations

$$\frac{dx^i}{da} = \xi^i(\bar{x},\bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x},\bar{u}), \tag{1.9}$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x^i, \quad \bar{u}^\alpha|_{a=0} = u^\alpha.$$

By definition of symmetry, the transformations (1.3) form a symmetry group G of the system (1.1) if the function $\bar{u} = \bar{u}(\bar{x})$ satisfies

$$E^\alpha(\bar{x},\bar{u},\bar{u}_{(1)},...,\bar{u}_{(k)}) = 0, \quad \sigma = 1,...,\bar{m}, \tag{1.10}$$

whenever the function $u = u(x)$ satisfies (1.1). The transformed derivatives $\bar{u}_{(1)},...,\bar{u}_{(k)}$ are found from (1.3) by using the formulae of change of variables in the derivatives, $D_i = D_i(f^j)\bar{D}_j$. Here

$$D_j = \frac{\partial}{x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + ...$$

is the total derivative operator with respect to x^i and $\bar{D}_j$ is likewise given in terms of the transformed variables. The transformations (1.3) together with the transformations on $\bar{u}_{(1)}$ form a group, $G^{[1]}$, called the first prolonged group which acts in the space of $(x, u, u_{(1)})$. Similarly, the prolonged groups $G^{[2]}, \cdots, G^{[k]}$ are obtained. The infinitesimal transformations of the prolonged groups are:

$$\bar{u}_i^\alpha \approx u_i^\alpha + a\zeta_i^\alpha(x, u, u_{(1)}),$$

$$\bar{u}_{ij}^\alpha \approx u_{ij}^\alpha + a\zeta_{ij}^\alpha(x, u, u_{(1)}, u_{(2)}),$$

$$\vdots \tag{1.11}$$

$$\bar{u}_{i_1 \dots i_k}^\alpha \approx u_{i_1 \dots i_k}^\alpha + a\zeta_{i_1 \dots i_k}^\alpha(x, u, u_{(1)}, \dots, u_{(k)}).$$

The functions $\zeta_i^\alpha(x, u, u_{(1)})$, $\zeta_{ij}^\alpha(x, u, u_{(1)}, u_{(2)})$ and $\zeta_{i_1 \dots i_k}^\alpha(x, u, u_{(1)}, \dots, u_{(k)})$, are given by

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j),$$

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{i_1 \dots i_k l}^\alpha D_j(\xi^l),$$

$$\vdots \tag{1.12}$$

$$\zeta_{i_1 \dots i_k}^\alpha = D_{i_k}(\zeta_{i_1 \dots i_{k-1}}^\alpha) - u_{i_1 \dots i_k l}^\alpha D_j(\xi^l),$$

and the generators of the prolonged groups are

$$X^{[1]} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x, u, u_{(1)})\frac{\partial}{\partial u_i^\alpha},$$

$$\vdots \tag{1.13}$$

$$X^{[k]} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x, u, u_{(1)})\frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta_{i_1 \dots i_k}^\alpha(x, u, \dots, u_{(k)})\frac{\partial}{\partial u_{i_1 \dots i_k}^\alpha}.$$

Definition 3 We say that a differential function $F(x, u, ..., u_{(p)})$, $p \geq 0$, is a pth-order differential invariant of a group G if

$$F(x, u, ..., u_{(p)}) = F(\bar{x}, \bar{u}, ..., \bar{u}_{(p)}),$$

i.e., if F is invariant under the prolonged group $G^{[p]}$. For $p = 0$, we write $u_{(0)} = u$ and $G^{[0]} = G$.

Theorem 3 A differential function $F(x, u, ..., u_{(p)})$, $p \geq 0$, is a pth order differential invariant of a group G if

$$X^{[p]}F = 0,$$

where $X^{[p]}$ is the pth prolongation of X and for $p = 0$, $X^{[0]} \equiv X$. The substitution of (1.6) and (1.11) into (1.10) gives rise to

$$E^{\sigma}(\bar{x}, \bar{u}, \bar{u}_{(1)}, ..., \bar{u}_{(k)}) \approx E^{\sigma}(x, u, u_{(1)}, ..., u_{(k)}) + a(X^{[k]}E^{\sigma}), \quad \sigma = 1, 2, ..., \bar{m}. \quad (1.14)$$

Thus, for invariance of (1.1) we require

$$X^{[k]}E^{\sigma}(x, u, u_{(1)}, ..., u_{(k)}) = 0, \quad \sigma = 1, 2, ..., \bar{m}, \quad (1.15)$$

whenever (1.1) is satisfied.

The converse is also true.

Theorem 4 The equations (1.15) define all infinitesimal symmetries of the system (1.1).

We call equations (1.15) the determining equations and write them as

$$X^{[k]}E^{\sigma}(x, u, u_{(1)}, ..., u_{(k)})|_{1.1} = 0, \quad \sigma = 1, ..., \bar{m}, \quad (1.16)$$

where $|_{(1.1)}$ means evaluated on the surface (1.1). They are linear homogeneous partial differential equations of order k for the unknown functions $\xi(x, u)$ and $\eta^{\alpha}(x, u)$ and are consequences of the prolongation formulae (1.12). In general, (1.16) decomposes into an overdetermined system of equations, that is, there are more equations than the $n + m$ unknowns ξ^i and η^{α}. One can use computer algebra programs to obtain and, in some cases, solve the determining equations.

The solutions of the determining equations form a vector space L and furthermore, if the generators

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$

satisfy the determining equations, their commutator $[X_1, X_2] = X_1X_2 - X_2X_1$,

$$[X_1, X_2] = \left(X_1(\xi_2^i) - X_2(\xi_1^i)\right)\frac{\partial}{\partial x^i} + \left(X_1(\eta_2^\alpha) - X_2(\eta_1^\alpha)\right)\frac{\partial}{\partial u^\alpha},$$

which obeys the properties of bilinearity, skew-symmetry and Jacobi's identity, is also a solution of the determining equations. Thus, the vector space L of all solutions of the determining equations froms a Lie algebra which generates a multi-parameter group admitted by (1.1).

1.3 Symmetries and Conservation Laws

In the study of differential equations, the concept of a conservation law, which is a mathematical formulation of the familiar physical laws of conservation of energy, conservation of momentum and so on, plays a very important role in the analysis of basic properties of the solutions (see for example, [8] and [14]). In this section we briefly comment on the relationship between symmetries and conservation laws.

Definition 4 A conserved vector of the differential equation $E^\sigma(x, u, ..., u_{(k)}) = 0$ is a tuple $T = (T^1, ..., T^n)$, $T^j = T^j(x, u, u_{(1)}, ..., u_{(k-1)})$, $j = 1, ..., n$, such that

$$D_i(T^i) = 0 \qquad (1.17)$$

is satisfied for all solutions of the differential equation.

The equation (1.17) is called a local conservation law.

1.3.1 Noether symmetries and Noether's theorem

In 1918, Emmy Noether proved the remarkable result: For Euler-Lagrange differential equations, to each Noether symmetry associated with a Lagrangian there corresponds a conservation law which can be determined explicitly by a formula [15]. The relationship between the Noether conserved vector components T^is and the Lie-Bäcklund symmetry operator X is given by (see [16])

$$X(T^i)+D_k(\xi^k)(T^i)-D_k(\xi^i)(T^k) = N^i(D_k(B^k))+D_k(\xi^i)(B^k)-D_k(\xi^k)(B^i)-X(B^i),$$

$$i = 1, ..., n. \quad (1.18)$$

Here N^i are the Noether operators, see [16]. We now ask ourself a question. Does the above relation apply to differential equations without a Lagrangian (like scalar evolution differential equations)? The answer is provided in the result below.

1.3.2 Arbitrary differential equations: Symmetry and conservation laws

The fundamental relationship between the Lie-Bäcklund symmetry generator X and the conserved vector T for a differential equation without a Lagrangian is given by (see [17])

$$X(T^i) + D_k(\xi^k)T^i - D_k(\xi^i)T^k = 0, \quad i = 1, ..., n. \quad (1.19)$$

If this relation holds, the generator X is said to be associated with the conservation law (1.17).

There are many uses of the above result [14]. We list a few of them: Firstly, the equations (1.19) relate T and X. These are simplifying conditions when one uses them together with (1.17) to construct conservation laws with known symmetry. Usually one utilises the direct method, viz., (1.17), without recourse to (1.19), in the absence of a Lagrangian.

Secondly, one can calculate symmetries associated with given conservation law by invoking (1.19).

Thirdly, the equations (1.19) can be invoked in the construction of Lagrangians for differential equations.

An application of the above result can be seen in [18]. The authors of [18] have used the invariance of a conservation law related to volume to obtain solutions for a problem in thin films.

Recently, it has been shown that for a system of partial differential equations, one can generate conservation laws from known ones using any Lie-Bäcklund symmetry operator of the system without having to make a conversion to a canonical Lie-Bäcklund symmetry operator. For more details on this see [19].

Chapter 2

Invariant Solutions of certain Soil Water Equations

2.1 Introduction

In this chapter we shall use Lie group analysis to obtain exact/asymptotic invariant solutions of soil water equations

$$C(\psi)\psi_t = (K(\psi)\psi_x)_x + (K(\psi)(\psi_z - 1))_z - S(\psi) \tag{2.1}$$

for some special forms of the functions $C(\psi)$, $K(\psi)$ and $S(\psi)$. For each case we shall look for solutions invariant under two-dimensional subalgebras of the symmetry Lie algebra.

We first describe the general algorithm due to Lie [20] and Ovsiannikov [21] for constructing invariant solutions.
We choose two operators

$$X_1 = \xi_1^0(t,x,z,u)\frac{\partial}{\partial t} + \xi_1^1(t,x,z,u)\frac{\partial}{\partial x} + \xi_1^2(t,x,z,u)\frac{\partial}{\partial z} + \eta_1(t,x,z,u)\frac{\partial}{\partial u}$$

and

$$X_2 = \xi_2^0(t,x,z,u)\frac{\partial}{\partial t} + \xi_2^1(t,x,z,u)\frac{\partial}{\partial x} + \xi_2^2(t,x,z,u)\frac{\partial}{\partial z} + \eta_2(t,x,z,u)\frac{\partial}{\partial u}$$

admitted by equation (2.1) such that they form a two-dimensional Lie algebra, that is, $[X_1, X_2] = \lambda_1 X_1 + \lambda_2 X_2$, where λ_1, λ_2 are constants, and

$$\text{rank}\begin{pmatrix} \xi_1^0 & \xi_1^1 & \xi_1^2 & \eta_1 \\ \xi_2^0 & \xi_2^1 & \xi_2^2 & \eta_2 \end{pmatrix} = 2.$$

Now under these conditions, the system

$$X_1 I = 0, \quad X_2 I = 0$$

has exactly two functionally independent solutions $I_1(t, x, z, u)$, $I_2(t, x, z, u)$ and the invariant solution has the form

$$I_2 = \phi(I_1). \tag{2.2}$$

We solve this for ψ and substitute into the corresponding equation (2.1). This will give us an ordinary differential equation for the function ϕ, which is then solved.

According to the group classification results (see [3]) the principal Lie algebra L_p, which is the Lie algebra of the Lie transformation group admitted by equation (2.1) for arbitrary functions $C(\psi)$, $K(\psi)$ and $S(\psi)$, is the three dimensional Lie algebra spanned by the operators which generate translations along t, x and z-axis, namely

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial z}$$

respectively.

2.2 Invariant solutions

We now obtain invariant solutions for certain soil water equations. We shall consider those cases in which the principal Lie algebra L_p extends by one or more operators.

2.2.1 Case 1

We first consider the case from Baikov and Khalique [5], that is, when $K(\psi) = 1$, $C(\psi) = \psi^{\sigma}$, where σ is an arbitrary constant and $S(\psi) = 0$.

In this case equation (1) has the form

$$\psi_t = \psi^{-\sigma} \left(\psi_{xx} + \psi_{zz} \right). \tag{2.3}$$

This equation admits a six-dimensional Lie algebra L_6 (see [3]) obtained by an extension of the principal Lie algebra L_p by the following three operators:

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$
$$X_5 = \sigma t\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial \psi},$$

and

$$X_6 = \sigma x\frac{\partial}{\partial x} + \sigma z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}.$$

We can now construct invariant solutions by considering two operators at a time.

Case 1.1. Let us start by considering the operators X_4 and X_5 and use them to construct invariant solutions. These two operators span a two-dimensional subalgebra L_2 of the algebra L_6 and have two functionally independent invariants. We first calculate a basis of invariants $I(t, x, z, \psi)$ by solving the system of linear first-order partial differential equations:

$$X_4 I = 0, \quad X_5 I = 0.$$

Since we have $[X_4, X_5] = 0$, the subalgebra L_2 is Abelian. Therefore we can solve the equations $X_4 I = 0, X_5 I = 0$ successively in any order. From $X_4 I = 0$, we have

$$\frac{dx}{z} = -\frac{dz}{x} = \frac{d\psi}{0} = \frac{dt}{0}.$$

First equation gives us

$$xdx + zdz = 0,$$

and intergrating, we obtain $x^2 + z^2 = C$, where C is a constant. Thus we have the following three functionally independent solutions

$$J_1 = x^2 + z^2, \quad J_2 = \psi \quad \text{and} \quad J_3 = t.$$

Rewriting the action of X_5 on the space of J_1, J_2 and J_3 by the formula

$$X_5 = X_5(J_1)\frac{\partial}{\partial J_1} + X_5(J_2)\frac{\partial}{\partial J_2} + X_5(J_3)\frac{\partial}{\partial J_3}$$

we obtain

$$X_5 = J_2\frac{\partial}{\partial J_2} + \sigma J_3\frac{\partial}{\partial J_3}.$$

Now from the second equation $X_5 I = 0$, we have

$$J_2\frac{\partial I}{\partial J_2} + \sigma J_3\frac{\partial I}{\partial J_3} = 0.$$

This gives us

$$\frac{dJ_2}{J_2} = \frac{dJ_3}{\sigma J_3}$$

and intergrating both sides, we obtain

$$\ln J_2 = \frac{1}{\sigma}\ln J_3 + \ln C$$

or

$$C = J_2 J_3^{-\frac{1}{\sigma}}. \tag{2.4}$$

Thus our first invariant, from (2.4), is given by

$$I_1 = J_2 J_2^{-\frac{1}{\sigma}} = \psi t^{-\frac{1}{\sigma}}.$$

The second invariant is

$$I_2 = x^2 + z^2.$$

Hence we have obtained two functionally independent invariants and so the invariant solution of equation (2.3) can be written as

$$I_1 = \phi(I_2),$$

that is,

$$\psi t^{\frac{-1}{\sigma}} = \phi(x^2 + z^2),$$

or

$$\psi = t^{\frac{1}{\sigma}}\phi(x^2 + z^2). \tag{2.5}$$

We now substitute the value of ψ from equation (2.5) into equation (2.3). For this we need to determine ψ_t, ψ_{xx} and ψ_{zz}. We obtain

$$\psi_t = \frac{1}{\sigma} t^{\frac{1}{\sigma}-1} \phi(x^2 + z^2),$$

$$\psi_{xx} = 2t^{\frac{1}{\sigma}} \left\{ \phi' + 2x^2 \phi'' \right\}$$

and

$$\psi_{zz} = 2t^{\frac{1}{\sigma}} \left(\phi' + 2z^2 \phi'' \right).$$

Here $'$ denotes the derivative of ϕ with respect to its argument. Substituting these values in (2.3), we obtain

$$\frac{1}{\sigma} t^{\frac{1}{\sigma}-1} \phi = \left(t^{\frac{1}{\sigma}} \phi \right)^{-\sigma} \{ 4\phi' t^{\frac{1}{\sigma}} + 4\phi''(x^2 + z^2) t^{\frac{1}{\sigma}} \}.$$

Simplyfying we have

$$\frac{1}{\sigma} t^{\frac{1}{\sigma}} \phi^{\sigma+1} = 4\phi' t^{\frac{1}{\sigma}} + 4\phi''(x^2 + z^2) t^{\frac{1}{\sigma}},$$

or

$$\frac{1}{\sigma} \phi^{\sigma+1} = 4(\xi\phi'' + \phi') \quad \text{where } \xi = (x^2 + z^2),$$

and finally we obtain

$$4\xi\phi'' + 4\phi' - \frac{1}{\sigma}\phi^{\sigma+1} = 0 \tag{2.6}$$

Equation (2.6) is a nonlinear second-order ordinary differential equation which is not easy to solve. Here we will find a particular solution of the form

$$\phi = A\xi^n. \tag{2.7}$$

Substituting into (2.6), we obtain

$$4\xi n(n-1)A\xi^{n-2} + 4nA\xi^{n-1} - \frac{1}{\sigma} A^{\sigma+1} \xi^{n(\sigma+1)} = 0$$

or

$$4An(n-1)\xi^{n-1} + 4nA\xi^{n-1} - \frac{1}{\sigma} A^{\sigma+1} \xi^{n(\sigma+1)} = 0. \tag{2.8}$$

In order for ξ to have the same power in the above equation, we must have

$$n - 1 = n(\sigma + 1)$$

$$n - 1 = n\sigma + n \quad \Rightarrow n = -\frac{1}{\sigma}$$

and we then obtain

$$4An(n-1) + 4nA - \frac{1}{\sigma}A^{\sigma+1} = 0$$

which gives us

$$A = \left(\frac{4}{\sigma}\right)^{\frac{1}{\sigma}}.$$

Hence from equation (2.6) we obtain

$$\phi = \left(\frac{4}{\sigma}\right)^{\frac{1}{\sigma}} \xi^{-\frac{1}{\sigma}}$$

and equation (2.5) gives us

$$\psi = t^{\frac{1}{\sigma}} \left(\frac{4}{\sigma}\right)^{\frac{1}{\sigma}} \left(\frac{1}{\xi}\right)^{\frac{1}{\sigma}}$$

or

$$\psi = \left(\frac{4t}{\sigma(x^2 + z^2)}\right)^{\frac{1}{\sigma}}$$

which is an invariant solution of equation (2.3).

Case 1.2

We now use the operators X_4 and X_6 to find the invariant solutions of equation (2.3). Using the above algorithm, in this case, the first equation $X_4 I = 0$ provides us with three functionally independent solutions $J_1 = x^2 + z^2, J_2 = \phi$ and $J_3 = t$. Now writing X_6 in the space of J_1, J_2 and J_3, we obtain

$$X_6 = 2\sigma J_1 \frac{\partial}{\partial J_2} - 2J_2 \frac{\partial}{\partial J_2}.$$

and using the second equation $X_6 I = 0$ we following two functionally independent solutions (invariants) are obtained:

$$I_1 = J_1^{\frac{1}{\sigma}} J_2 \equiv \left(x^2 + z^2\right)^{\frac{1}{\sigma}} \psi, \quad I_2 = J_3 \equiv t.$$

Thus the invariant solution is given by $I_1 = \phi(I_2)$, that is

$$\left(x^2+z^2\right)^{\frac{1}{\sigma}}\psi = \phi(t)$$

or

$$\psi = \left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi(t). \tag{2.9}$$

Differentiating ψ with respect to t, and twice with respect to x and z, we obtain:

$$\psi_t = \left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi',$$

$$\psi_{xx} = -\frac{2}{\sigma}\phi\left\{\left(x^2+z^2\right)^{-\frac{1}{\sigma}-1} + 2x^2\left(-\frac{1}{\sigma}-1\right)\left(x^2+z^2\right)^{-\frac{1}{\sigma}-2}\right\}$$

and

$$\psi_{zz} = -\frac{2}{\sigma}\phi\left\{\left(x^2+z^2\right)^{-\frac{1}{\sigma}-1} + 2z^2\left(-\frac{1}{\sigma}-1\right)\left(x^2+z^2\right)^{-\frac{1}{\sigma}-2}\right\}.$$

Substituting these values of ψ_t, ψ_{xx} and ψ_{zz} in equation (2.3), we obtain

$$\left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi' = -\frac{2}{\sigma}\left(x^2+z^2\right)\phi^{-\sigma+1}\left\{2\left(x^2+z^2\right)^{-\frac{1}{\sigma}-1} + 2(x^2+z^2)\left(-\frac{1}{\sigma}-1\right)\left(x^2+z^2\right)^{-\frac{1}{\sigma}-2}\right\}.$$

This simplifies to

$$\left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi' = -\frac{2}{\sigma}\left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi^{-\sigma+1}\left\{2+2\left(-\frac{1}{\sigma}-1\right)\right\}$$

or

$$\left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi' = -\frac{4}{\sigma}\left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi^{-\sigma+1}\left\{-\frac{1}{\sigma}\right\}.$$

Further simplification gives

$$\left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi' = \frac{4}{\sigma^2}\left(x^2+z^2\right)^{-\frac{1}{\sigma}}\phi^{-\sigma+1}$$

or

$$\phi' = \frac{4}{\sigma^2}\phi^{-\sigma+1}. \tag{2.10}$$

Re-writing equation (2.10), we have

$$\frac{d\phi}{dt} = \frac{4}{\sigma^2}\phi^{1-\sigma}$$

This is a first-order separable equation. Separating the variables, we obtain

$$\phi^{\sigma-1}d\phi = \frac{4}{\sigma^2}dt$$

Integrating gives us

$$\frac{\phi^{\sigma}}{\sigma} = \frac{4}{\sigma^2}t + C_1$$

or

$$\phi^{\sigma} = \frac{4}{\sigma}(t + C).$$

Thus we obtain

$$\phi = \left(\frac{4}{\sigma}(t + C)\right)^{\frac{1}{\sigma}}$$

and substituting this in equation (2.9) gives

$$\psi = \left(\frac{\sigma(x^2 + z^2)}{4(t + C)}\right)^{-\frac{1}{\sigma}}$$

which is an invariant solution of equation (2.3).

Case 1.3

Likewise using operators X_5 and X_6 one can obtain an invariant solution of equation (2.3).

Case 1.4

We now construct invariant solutions under the operator X_6 and a combination of the operators X_1 and X_5, that is $X_1 + X_5$, where

$$X_1 + X_5 = (1 + \sigma t)\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial \psi}.$$

The operators $X_1 + X_5$ and X_6 span a two-dimensional subalgebra L_2 of the algebra L_6 and have two functionally independent invariants. To find these we calculate a basis of invariants $I(t, x, z, \psi)$ by solving the two equations $(X_1 + X_5)I = 0$ and $X_6 I = 0$. Since $[X_1 + X_5, X_6] = 0$, the subalgebra is Abelian. Therefore we can solve the equations in any order. The second equation gives us three functionally independent solutions. We obtain these as follows: The equation $X_6 I = 0$ implies that

$$\frac{dx}{\sigma x} = \frac{dz}{\sigma z} = -\frac{d\psi}{-2\psi} = \frac{dt}{0}.$$

We first consider

$$\frac{dx}{\sigma x} = \frac{dz}{\sigma z}.$$

Integrating, we obtain

$$\ln x - \ln z = \ln C_1$$

or

$$\frac{x}{z} = C_1 \equiv J_1$$

Secondly

$$\frac{dz}{\sigma z} = \frac{d\psi}{-2\psi}$$

gives

$$\frac{2}{\sigma}\frac{dz}{z} = \frac{d\psi}{--\psi}.$$

Integrating, we obtain

$$\ln z^{\frac{2}{\sigma}} = -\ln\psi + \ln C_2$$

or

$$z^{\frac{2}{\sigma}}\psi = C_2 \equiv J_2.$$

Thus we have obtained three functionally independent solutions J_1, J_2 and J_3. Hence the common solution $I(t, x, z, \psi)$ of our system is defined as a function of J_1, J_2 and J_3 only. Therefore we rewrite the action of $X_1 + X_5$ on the space of J_1, J_2 and J_3 by the formula

$$X_1 + X_5 = (X_1 + X_5)(J_1)\frac{\partial}{\partial J_1} + (X_1 + X_5)(J_2)\frac{\partial}{\partial J_2} + (X_1 + X_5)(J_3)\frac{\partial}{\partial J_3}$$

to obtain

$$X_1 + X_5 = J_2\frac{\partial}{\partial J_2} + (1 + \sigma J_3)\frac{\partial}{\partial J_3}.$$

Thus from the second equation $(X_1 + X_5)I = 0$, we obtain

$$\frac{dJ_2}{J_2} = \frac{dJ_3}{1 + \sigma J_3}.$$

Integrating, gives us

$$\ln J_2 = \frac{1}{\sigma}\ln(1 + \sigma J_3) + \ln C_1$$

or

$$C_1 = J_2(1+\sigma J_3)^{\frac{-1}{\sigma}}$$

Thus the two functionally independent solutions (invariants) are:

$$I_1 = J_2(1+\sigma J_3)^{\frac{-1}{\sigma}} \equiv z^{\frac{2}{\sigma}}\psi(1+\sigma t)^{\frac{-1}{\sigma}},$$

and

$$I_2 = J_1 \equiv \frac{x}{z}.$$

Consequently, the invariant solution is given by $I_1 = \phi(I_2)$, that is

$$z^{\frac{2}{\sigma}}\psi(1+\sigma t)^{\frac{-1}{\sigma}} = \phi(\frac{x}{z})$$

or

$$\psi = z^{\frac{-2}{\sigma}}(1+\sigma t)^{\frac{1}{\sigma}}\ \phi(\frac{x}{z}). \tag{2.11}$$

We now find ψ_t, ψ_{xx} and ψ_{zz}.

$$\psi_t = z^{-\frac{2}{\sigma}}\,(1+\sigma t)^{\frac{1}{\sigma}-1}\,\phi,$$

$$\psi_{xx} = z^{-\frac{2}{\sigma}-2}\,(1+\sigma t)^{\frac{1}{\sigma}}\,\phi''$$

and

$$\psi_{zz} = (1+\sigma t)^{\frac{1}{\sigma}} z^{-\frac{2}{\sigma}-2}\left\{\frac{2}{\sigma}\left(\frac{2}{\sigma}+1\right)\phi + 2\left(\frac{2}{\sigma}+1\right)\frac{x}{z}\phi' + \frac{x^2}{z^2}\phi''\right\}.$$

Substituting the above values of the derivatives of ψ in equation (2.3), we obtain

$$z^{-\frac{2}{\sigma}}\,(1+\sigma t)^{\frac{1}{\sigma}-1}\,\phi = z^2\,(1+\sigma t)^{-1}\,\phi^{-\sigma}\left\{z^{-\frac{2}{\sigma}-2}\,(1+\sigma t)^{\frac{1}{\sigma}}\,\phi'' + (1+\sigma t)^{\frac{1}{\sigma}} z^{-\frac{2}{\sigma}-2}\left[\frac{2}{\sigma}\right.\right.$$

$$\left.\left.\left(\frac{2}{\sigma}+1\right)\phi + 2\left(\frac{2}{\sigma}+1\right)\frac{x}{z}\phi' + \frac{x^2}{z^2}\phi''\right]\right\}.$$

This simplifies to

$$\left(1+\xi^2\right)\phi'' + 2\left(\frac{2}{\sigma}+1\right)\xi\phi' + \frac{2}{\sigma}\left(\frac{2}{\sigma}+1\right)\phi - \phi^{\sigma+1}, \quad \text{where} \quad \xi = \frac{x}{z}. \tag{2.12}$$

It can be seen that

$$\phi_0 = \left[\frac{2}{\sigma}\left(\frac{2}{\sigma}+1\right)\right]^{\frac{1}{\sigma}}$$

is a constant solution of equation (2.12). We now obtain an approximate solution of equation (2.12) near ϕ_0. By letting $\phi = \phi_0 + \phi_1$ we linearize equation (2.12) near the constant solution ϕ_0. We obtain

$$(1+\xi^2)\phi_1'' + \left(2+\frac{4}{\sigma}\right)\xi\phi_1' - 2\left(\frac{2}{\sigma}+1\right)\phi_1 = 0$$

whose general solution is given by (see for example [22])

$$\phi_1 = C_1\phi_{11} + C_2\phi_{12}$$

where C_1 and C_2 are arbitrary constants,

$$\phi_{11} = F\left(-\frac{a}{2}+\frac{2}{\sigma}, \frac{a}{2}+\frac{2}{\sigma}+\frac{1}{2}; \frac{1}{2}; -\xi^2\right)$$

and

$$\phi_{12} = \xi F\left(-\frac{a}{2}+\frac{2}{\sigma}+\frac{1}{2}, \frac{a}{2}+\frac{2}{\sigma}+1; \frac{3}{2}; -\xi^2\right).$$

Here F is a hypergeometric function and a satisfies the quadratic equation

$$\alpha^2 + \alpha - \frac{4}{\sigma^2} - \frac{6}{\sigma} - 2 = 0.$$

Hence the approximate invariant solution of equation (2.3) is given by

$$\psi = z^{\frac{-2}{\sigma}}(1+\sigma t)^{\frac{1}{\sigma}}\left[\left(\frac{2}{\sigma}\left(\frac{2}{\sigma}+1\right)\right)^{\frac{1}{\sigma}} + C_1\phi_{11} + C_2\phi_{12}\right].$$

2.2.2 Case 2

In our last example of constructing invariants solutions we consider equation (2.1) when $K(\psi)$=1, $C(\psi) = 1$ and $S(\psi) = -\psi^2$, that is

$$\psi_t = \psi_{xx} + \psi_{zz} + \psi^2. \tag{2.13}$$

Group classification results in [4] tells us that equation (2.13) admits a five-dimensional Lie algebra L_5 obtained by an extension of the principal Lie algebra L_p by the following operators:

$$X_4 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

and

$$X_5 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}.$$

Let us now construct invariant solutions under the operators $X_1 + X_5$ and X_4, where

$$X_1 + X_5 = (1+2t)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z} - 2\psi\frac{\partial}{\partial \psi}.$$

Since $[X_4, X_1 + X_5] = 0$, the subalgebra L_2 is Abelian. Therefore we can solve the equations $X_4 I = 0$, $X_5 I = 0$ successively in any order. The first equation provides us with three functionally independent solutions $J_1 = x^2 + z^2$, $J_2 = \psi$ and $J_3 = t$. Hence the common solution $I(t, x, z, \psi)$ of the system is defined as a function of J_1, J_2 and J_3 only. Writing the action of $X_1 + X_5$ on the space of J_1, J_2 and J_3 we obtain

$$X_1 + X_5 = (X_1 + X_5)(J_1)\frac{\partial}{\partial J_1} + (X_1 + X_5)(J_2)\frac{\partial}{\partial J_2} + (X_1 + X_5)(J_3)\frac{\partial}{\partial J_3}$$

or

$$X_1 + X_5 = 2J_1\frac{\partial}{\partial J_1} - 2J_2\frac{\partial}{\partial J_2} + (1+2J_3)\frac{\partial}{\partial J_3}.$$

Consequently, the second equation $(X_1 + X_5)I = 0$ yields

$$\frac{dJ_1}{2J_1} = \frac{dJ_2}{-2J_2} = \frac{dJ_3}{(1+2J_3)}.$$

From the first equation, we obtain

$$\ln J_1 + \ln J_2 = \ln C, \quad \text{or} \quad J_1 J_2 = C.$$

Thus

$$I_1 = (x^2 + z^2)\psi \tag{2.14}$$

and from the second equation, we obtain

$$\frac{dJ_1}{2J_1} = \frac{dJ_3}{1+2J_3}.$$

Integrating, we obtain

$$\frac{1}{2}\ln J_1 - \frac{1}{2}\ln(1+2J_3) = \ln C_1$$

or

$$\frac{J_1}{1+2J_3} = C$$

Hence

$$I_2 = \frac{(x^2+z^2)}{1+2t}. \tag{2.15}$$

Thus the invariant solution is given by

$$I_1 = \phi(I_2),$$

that is,

$$(x^2+z^2)\psi = \phi\left(\frac{x^2+z^2}{1+2t}\right)$$

or

$$\psi = \frac{1}{x^2+z^2}\phi(\xi), \quad \text{where} \quad \xi = \frac{x^2+z^2}{1+2t}.$$

Now

$$\psi_t = -\frac{2\phi'}{(1+2t)^2},$$

$$\psi_{xx} = \frac{4x^2\phi''}{(1+2t)^2(x^2+z^2)} + \left\{\frac{2}{(x^2+z^2)(1+2t)} - \frac{8x^2}{(1+2t)(x^2+z^2)^2}\right\}\phi'$$
$$+\left\{-\frac{2}{(x^2+z^2)^2} + \frac{8x^2}{(x^2+z^2)^3}\right\}\phi$$

and

$$\psi_{zz} = \frac{4z^2\phi''}{(1+2t)^2(x^2+z^2)} + \left\{\frac{2}{(1+2t)(x^2+z^2)} - \frac{8z^2}{(1+2t)(x^2+z^2)^2}\right\}\phi'$$
$$+\left\{-\frac{2}{(x^2+z^2)^2} + \frac{8z^2}{(x^2+z^2)^3}\right\}\phi.$$

Substituting the above in equation (2.13), we obtain

$$\frac{-2\phi'}{(1+2t)^2} = \frac{4(x^2+z^2)\phi''}{(1+2t)^2(x^2+z^2)} + \left\{\frac{4}{(1+2t)(x^2+z^2)} - \frac{8(x^2+z^2)}{(1+2t)(x^2+z^2)^2}\right\}\phi'$$
$$+\left\{-\frac{4}{(x^2+z^2)^2} + \frac{8(x^2+z^2)}{(x^2+z^2)^3}\right\}\phi + \frac{\phi^2}{(x^2+z^2)^2}$$

which simplifies to

$$4\xi^2\phi'' + (2\xi^2 - 4\xi)\phi' + 4\phi + \phi^2 = 0.$$

This is a second-order nonlinear ordinary differential equation. We note that $\phi_0 = -4$ is a constant solution of this equation. By letting $\phi = \phi_0 + \phi_1$ we linearize the equation and obtain

$$\phi_1'' + \left(\frac{1}{2} - \frac{1}{\xi}\right)\phi_1' - \frac{1}{\xi^2}\phi_1 = 0. \tag{2.16}$$

If we let

$$\phi_1 = ye^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi} \tag{2.17}$$

then

$$\phi_1' = y'e^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi} + ye^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi}\left(-\frac{1}{4}+\frac{1}{2\xi}\right)$$

and

$$\phi_1'' = y''e^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi} + 2y'e^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi}\left(-\frac{1}{4}+\frac{1}{2\xi}\right) + ye^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi}\left(-\frac{1}{4}+\frac{1}{2\xi}\right)^2$$
$$+ye^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi}\left(-\frac{1}{2\xi^2}\right).$$

Substituting these values in equation (2.16), we obtain

$$y'' + 2y'\left(\frac{1}{2\xi}-\frac{1}{4}\right) + y\left(\frac{1}{2\xi}-\frac{1}{4}\right)^2 - \frac{y}{2\xi^2} + \left(\frac{1}{2}-\frac{1}{\xi}\right)\left\{y' + y\left(\frac{1}{2\xi}-\frac{1}{4}\right)\right\} - \frac{1}{\xi^2}y = 0$$

or

$$y'' + y\left\{\left(\frac{1}{2\xi}-\frac{1}{4}\right)^2 - \frac{1}{2\xi^2} + \left(\frac{1}{2}-\frac{1}{\xi}\right)\left(\frac{1}{2\xi}-\frac{1}{4}\right) - \frac{1}{\xi^2}\right\} = 0$$

which gives us

$$y'' + y\left\{\left(\frac{1}{16}+\frac{1}{4\xi^2}-\frac{1}{4\xi}-\frac{1}{2\xi^2}-\frac{1}{8}+\frac{1}{4\xi}+\frac{1}{4\xi}-\frac{1}{2\xi^2}-\frac{1}{\xi^2}\right)\right\} = 0.$$

Simplifying the above equation, we get

$$y'' + y\left\{-\frac{1}{16}+\frac{1}{4\xi}-\frac{7}{4\xi^2}\right\} = 0$$

This equation can be rewritten in the form

$$y'' = q(\xi)y \tag{2.18}$$

where

$$q(\xi) = \frac{1}{16} - \frac{1}{4\xi} + \frac{7}{4\xi^2}.$$

The Liouville-Green approximation for the general solution of equation (2.18) is given by (see for example [22])

$$y = Aq^{\frac{-1}{4}}e^{\int q^{\frac{1}{2}}d\xi} + Bq^{\frac{-1}{4}}e^{-\int q^{\frac{1}{2}}d\xi},$$

where A and B are arbitrary constants.Therefore an approximate invaraint solution of equation (2.13) is given by

$$\psi = \frac{1}{x^2+z^2}\left[-4+e^{-\frac{1}{2}\int\left(\frac{1}{2}-\frac{1}{\xi}\right)d\xi}\left\{Aq^{\frac{-1}{4}}e^{\int q^{\frac{1}{2}}d\xi}+Bq^{\frac{-1}{4}}e^{-\int q^{\frac{1}{2}}d\xi}\right\}\right].$$

Chapter 3

Conservation Laws for Equations Related to Soil Water Equations

In this chapter we obtain all nontrivial conservation laws for a class of (2+1) nonlinear evolution partial differential equations which are related to the soil water equations (see [23]).

We first note that the soil water equation

$$C(\psi)\psi_t = (K(\psi)\psi_x)_x + (K(\psi)(\psi_z - 1))_z - S(\psi), \tag{3.1}$$

can be rewritten as (see [3])

$$u_t = (k(u)u_x)_x + (k(u)u_z)_z + l(u)u_z + p(u), \tag{3.2}$$

where

$$u = \int C(\psi)d\psi, \; k(u) = K(\psi)/C(\psi), \; l(u) = -K'(\psi)/C(\psi), \; p(u) = -S(\psi).$$

The group classification of this equation with respect to admitted point transformation groups was performed in [24]. Group classification of equation (3.2) when $l(u) = 0$ with respect to Lie point symmetries was given in [25, 26] (see also [27] and [4], Vol. 1, Section 10.7).

Conservation laws were obtained for some special cases of (3.2) in [6] and symmetries were associated with them in the sense of [17].

Here we obtain all nontrivial conservation laws for equation (3.2). We list all the classes of equations (3.2) which admit point symmetries and for which nontrivial conservation laws exist.

We construct a conservation law

$$D_tT^1 + D_xT^2 + D_zT^3 = 0 \tag{3.3}$$

on the solutions of (3.2). See [23]. This gives us

$$\frac{\partial T^1}{\partial t} + k'(u_x^2 + ku_{xx} + ku_{zz} + lu_z + p)\frac{\partial T^1}{\partial u} + u_{tt}\frac{\partial T^1}{\partial u_t} + u_{tx}\frac{\partial T^1}{\partial u_x}$$

$$+u_{tz}\frac{\partial T^1}{\partial u_z} + \frac{\partial T^2}{\partial x} + u_x\frac{\partial T^2}{\partial u} + u_{xt}\frac{\partial T^2}{\partial u_t} + u_{xx}\frac{\partial T^2}{\partial u_x} + u_{xz}\frac{\partial T^2}{\partial u_z}$$

$$+\frac{\partial T^3}{\partial z} + u_z\frac{\partial T^3}{\partial u} + u_{zt}\frac{\partial T^3}{\partial u_t} + u_{xz}\frac{\partial T^3}{\partial u_x} + u_{zz}\frac{\partial T^3}{\partial u_z} = 0.$$

The separation of the second-order partial derivatives of u in the determining equation of (3.3) results in the following system of equations:

$$u_{tt} \quad : \frac{\partial T^1}{\partial u_t} = 0, \tag{3.4}$$

$$u_{xt} \quad : \frac{\partial T^1}{\partial u_x} + \frac{\partial T^2}{\partial u_t} = 0, \tag{3.5}$$

$$u_{zt} \quad : \frac{\partial T^1}{\partial u_z} + \frac{\partial T^3}{\partial u_t} = 0, \tag{3.6}$$

$$u_{zx} \quad : \frac{\partial T^2}{\partial u_z} + \frac{\partial T^3}{\partial u_x} = 0, \tag{3.7}$$

$$u_{xx} \quad : k(u)\frac{\partial T^1}{\partial u} + \frac{\partial T^2}{\partial u_x} = 0, \tag{3.8}$$

$$u_{zz} \quad : k(u)\frac{\partial T^1}{\partial u} + \frac{\partial T^3}{\partial u_z} = 0, \tag{3.9}$$

$$\text{rest} : \frac{\partial T^1}{\partial t} + \frac{\partial T^1}{\partial u}(k'u_x^2 + k'u_z^2 + l(u)u_z + p(u)) \tag{3.10}$$
$$+\frac{\partial T^2}{\partial x} + \frac{\partial T^2}{\partial u}u_x + \frac{\partial T^3}{\partial z} + \frac{\partial T^3}{\partial u}u_z = 0.$$

Equation (3.4) gives us $T^1 = T^1(t, x, z, u, u_x, u_z)$

Differentiating (3.5) with respect to u_t, we obtain

$$\frac{\partial^2 T^2}{\partial u_t^2} = 0$$

Integrating, we get

$$T^2 = \alpha(t, x, z, u, u_x, u_z)u_t + \beta(t, x, z, u, u_x, u_z).$$

Equation (3.5) now gives us

$$\frac{\partial T^1}{\partial u_x} + \alpha(t, x, z, u, u_x, u_z) = 0, \tag{3.11}$$

and equation (3.8) gives us

$$k(u)\frac{\partial T^1}{\partial u} + \frac{\partial \alpha}{\partial u_x}u_t + \frac{\partial \beta}{\partial u_x} = 0.$$

Spliting on u_t, we obtain

$$u_t : \frac{\partial \alpha}{\partial u_x} = 0 \tag{3.12}$$

$$u_t^0 : k(u)\frac{\partial T^1}{\partial u} + \frac{\partial \beta}{\partial u_x} = 0. \tag{3.13}$$

From (3.12) we obtain

$$\alpha = \alpha(t, x, z, u, u_z),$$

and equation (3.11) gives us

$$\frac{\partial T^1}{\partial u_x} = -\alpha.$$

Integrating we obtain

$$T^1 = -\alpha(t,x,z,u,u_z)u_x + \gamma(t,x,z,u,u_z)$$

and from equation (3.13), we have

$$k(u)\left\{-\frac{\partial\alpha}{\partial u}u_x + \frac{\partial\gamma}{\partial u}\right\} + \frac{\partial\beta}{\partial u_x} = 0.$$

This gives

$$\frac{\partial\beta}{\partial u_x} = k(u)\frac{\partial\alpha}{\partial u}u_x - k(u)\frac{\partial\gamma}{\partial u}$$

Integrating with respect to u_x, we obtain

$$\beta = k(u)\alpha_u\frac{u_x^2}{2} - k(u)\gamma_u u_x + \delta(t,x,z,u,u_z).$$

Thus

$$T^1 = -\alpha(t,x,z,u,u_z)u_x + \gamma(t,x,z,u,u_z)$$

$$T^2 = \alpha(t,x,z,u,u_z)u_t + k(u)\alpha_u\frac{u_x^2}{2} - k(u)\gamma_u u_x + \delta(t,x,z,u,u_z).$$

Equation (3.6) gives us

$$-\frac{\partial\alpha}{\partial u_z}u_x + \frac{\partial\gamma}{\partial u_z} + \frac{\partial T^3}{\partial u_t} = 0.$$

Integrating with respect to u_t, we obtain

$$T^3 = \frac{\partial\alpha}{\partial u_z}u_x u_t - \frac{\partial\gamma}{\partial u_z}u_t + \tau(t,x,z,u,u_x,u_z).$$

Equation (3.9) also gives us

$$k(u)\left[-\frac{\partial\alpha}{\partial u}u_x + \frac{\partial\gamma}{\partial u}\right] + \frac{\partial^2\alpha}{\partial u_z^2}u_x u_t - \frac{\partial^2\gamma}{\partial u_z^2}u_t + \frac{\partial\tau}{\partial u_z} = 0.$$

Splitting on u_t, we have

$$u_t : \frac{\partial^2\alpha}{\partial u_z^2}u_x - \frac{\partial^2\gamma}{\partial u_z^2} = 0 \qquad (3.14)$$

$$\text{rest} : k(u)[-\alpha_u u_x + \gamma_u] + \frac{\partial \tau}{\partial u_z} = 0. \tag{3.15}$$

Splitting (3.14) on u_x, we obtain

$$u_x : \frac{\partial^2 \alpha}{\partial u_z^2} = 0$$

$$u_x^0 : \frac{\partial^2 \gamma}{\partial u_z^2} = 0.$$

Integrating the above equations we obtain

$$\alpha = a_1(t, x, z, u)u_z + a_2(t, x, z, u)$$

$$\gamma = b_1(t, x, z, u)u_z + b_2(t, x, z, u).$$

Equation (3.15) now becomes

$$-k(u)\left\{\frac{\partial a_1}{\partial u}\ u_z + \frac{\partial a_2}{\partial u}\right\}u_x + k(u)\left\{\frac{\partial b_1}{\partial u}\ u_z + \frac{\partial b_2}{\partial u}\right\} + \frac{\partial \tau}{\partial u_z} = 0.$$

Integrating with respect to u_z, we obtain

$$\tau = k(u)\frac{\partial a_1}{\partial u}\ u_x\ \frac{u_z^2}{2} + k(u)\frac{\partial a_2}{\partial u}\ u_x\ u_z - k(u)\frac{\partial b_1}{\partial u}\ \frac{u_z^2}{2} - k(u)\frac{\partial b_2}{\partial u}u_z + \xi(t, x, z, u, u_x).$$

So

$$T^1 = -[a_1(t, x, z, u)u_z + a_2(t, x, z, u)]u_x + b_1(t, x, z, u)u_z + b_2(t, x, z, u),$$

$$T^2 = [a_1(t, x, z, u)u_z + a_2(t, x, z, u)]u_t + k(u)\frac{u_x^2}{2}\left[\frac{\partial a_1}{\partial u}u_z + \frac{\partial a_2}{\partial u}\right] - k(u)u_x\left[\frac{\partial b_1}{\partial u}u_z + \frac{\partial b_2}{\partial u}\right] + \delta(t, x, z, u, u_x),$$

$$T^3 = u_t u_x[a_1(t, x, z, u)] - u_t[b_1(t, x, z, u)] + k(u)\frac{\partial a_1}{\partial u}\ u_x\ \frac{u_z^2}{2} - k(u)\frac{\partial b_1}{\partial u}\ \frac{u_z^2}{2} + k(u)\frac{\partial a_2}{\partial u}\ u_x\ u_z - k(u)\frac{\partial b_2}{\partial u}\ u_z + \xi(t, x, z, u, u_x).$$

By equation (3.7), we obtain

$$a_1 u_t + k\ \frac{u_x^2}{2}\ \frac{\partial a_1}{\partial u} - k\ u_x\ \frac{\partial b_1}{\partial u} + \frac{\partial \delta}{\partial u_x} + a_1 u_t + k\frac{\partial a_1}{\partial u}\ \frac{u_z^2}{2} + k\frac{\partial a_2}{\partial u}\ u_z + \frac{\partial \xi}{\partial u_x} = 0.$$

Splitting the above equation on u_t, we obtain

$$u_t : a_1 = 0$$

$$u_t^0 : -ku_x \frac{\partial b_1}{\partial u} + \frac{\partial \delta}{\partial u_z} + k\,\frac{\partial a_2}{\partial u}\,u_z + \frac{\partial \xi}{\partial u_x} = 0. \tag{3.16}$$

Rewriting out T^1, T^2 and T^3 we have

$$T^1 = -a_2(t,x,u,z)u_x + b_1(t,x,z,u)u_z + b_2(t,x,z,u),$$

$$T^2 = a_2(t,x,z,u)u_t + k(u)\,\frac{u_x^2}{2}\,\frac{\partial a_2}{\partial u} - k(u)u_x\left\{\frac{\partial b_1}{\partial u}u_z + \frac{\partial b_2}{\partial u}\right\} + \delta(t,x,z,u,u_z),$$

$$T^3 = -b_1(t,x,z,u)u_t - k(u)\,\frac{\partial b_1}{\partial u}\,\frac{u_z^2}{2} + k(u)\,\frac{\partial a_2}{\partial u}\,u_xu_z - k(u)\,\frac{\partial b_2}{\partial u}u_z + \xi(t,x,z,u,u_x).$$

We now use equation (3.10). This gives us

$$-\frac{\partial a_2}{\partial t}\,u_x + \frac{\partial b_1}{\partial t}\,u_z + \frac{\partial b_2}{\partial t} + k'\,u_x^2\left(-\frac{\partial a_2}{\partial u}u_x + \frac{\partial b_1}{\partial u}u_z + \frac{\partial b_2}{\partial u}\right) + k'\,u_z^2\left(-\frac{\partial a_2}{\partial u}u_x + \frac{\partial b_1}{\partial u}u_z + \frac{\partial b_2}{\partial u}\right)$$

$$+l(u)\,u_z\left(-\frac{\partial a_2}{\partial u}u_x + \frac{\partial b_1}{\partial u}u_z + \frac{\partial b_2}{\partial u}\right) + p(u)\left(-\frac{\partial a_2}{\partial u}u_x + \frac{\partial b_1}{\partial u}u_z + \frac{\partial b_2}{\partial u}\right) + \frac{\partial a_2}{\partial x}u_t$$

$$+k(u)\frac{u_x^2}{2}\,\frac{\partial^2 a_2}{\partial u \partial x} - ku_xu_z\,\frac{\partial^2 b_1}{\partial x \partial u} - ku_x\,\frac{\partial^2 b_2}{\partial x \partial u} + \frac{\partial \delta}{\partial x} + u_x\left[\frac{\partial a_2}{\partial u}u_t + k'\frac{u_x^2}{2}\,\frac{\partial a_2}{\partial u}\right]$$

$$+u_x\left[k(u)\,\frac{u_x^2}{2}\,\frac{\partial^2 a_2}{\partial u^2} - k'u_x\left(\frac{\partial b_1}{\partial u}u_z + \frac{\partial b_2}{\partial u}\right) - ku_x\left(\frac{\partial^2 b_1}{\partial u^2}\,u_z + \frac{\partial^2 b_2}{\partial u^2}\right) + \frac{\partial \delta}{\partial u}\right]$$

$$-\frac{\partial b_1}{\partial z}u_t - k\,\frac{\partial^2 b_1}{\partial z \partial u}\,\frac{u_z^2}{2} + k(u)\,\frac{\partial^2 a_2}{\partial z \partial u}\,u_xu_z - k\,\frac{\partial^2 b_2}{\partial z \partial u}\,u_z + \frac{\partial \xi}{\partial z}$$

$$+u_z\left[-\frac{\partial b_1}{\partial u}\,u_t - k'\,\frac{\partial b_1}{\partial u}\,\frac{u_z^2}{2} - k\,\frac{\partial^2 b_1}{\partial u^2}\,\frac{u_z^2}{2} + k'\,\frac{\partial a_2}{\partial u}\,u_xu_z + k\,\frac{\partial^2 a_2}{\partial u^2}\,u_xu_z\right]$$

$$+u_z\left[-k'\frac{\partial b_2}{\partial u}\,u_z - k\frac{\partial^2 b_2}{\partial u^2}u_z + \frac{\partial \xi}{\partial u}\right] = 0.$$

Splitting the above equation, we obtain

$$u_t : \frac{\partial a_2}{\partial x} + u_x\frac{\partial a_2}{\partial u} - \frac{\partial b_1}{\partial z} - u_z\frac{\partial b_1}{\partial u} = 0,$$

and splitting this equation gives us

$$u_x : \frac{\partial a_2}{\partial u} = 0$$

$$u_z : \frac{\partial b_1}{\partial u} = 0$$

$$\text{rest} : \frac{\partial a_2}{\partial x} - \frac{\partial b_1}{\partial z} = 0, \tag{3.17}$$

and rewriting the above equation yields

$$-\frac{\partial a_2}{\partial t}u_x + \frac{\partial b_1}{\partial t}u_z + \frac{\partial b_2}{\partial t} + k'\ u_x^2\ \frac{\partial b_2}{\partial u} + k'\ u_z^2\ \frac{\partial b_2}{\partial u} + l(u)u_z\frac{\partial b_2}{\partial u}$$

$$+p(u)\frac{\partial b_2}{\partial u} - k\ u_x\ \frac{\partial^2 b_2}{\partial x \partial u} + \frac{\partial \delta}{\partial x} - k'\ u_x^2\ \frac{\partial b_2}{\partial u} - k\ u_x^2\ \frac{\partial^2 b_2}{\partial u^2}$$

$$+u_x\frac{\partial \delta}{\partial u} - k\ \frac{\partial^2 b_2}{\partial z \partial u}\ u_z + \frac{\partial \xi}{\partial z} - k'\ \frac{\partial b_2}{\partial u}\ u_z^2 - k\ \frac{\partial^2 b_2}{\partial u^2}\ u_z^2 + u_z\frac{\partial \xi}{\partial u} = 0. \tag{3.18}$$

Equation (3.16) then becomes

$$\frac{\partial \delta}{\partial u_z} + \frac{\partial \xi}{\partial u_x} = 0. \tag{3.19}$$

Differentiating equation (3.19) with respect to u_x, we obtain

$$\frac{\partial^2 \xi}{\partial u_x^2} = 0.$$

This gives us

$$\xi = c_1(t, x, z, u)u_x + c_2(t, x, z, u)$$

and substituting back into equation (3.18), we obtain

$$-\frac{\partial a_2}{\partial t}u_x + \frac{\partial b_1}{\partial t}u_z + \frac{\partial b_2}{\partial t} + l(u)\ u_z\frac{\partial b_2}{\partial u} + p(u)\frac{\partial b_2}{\partial u} - k\ u_x\ \frac{\partial^2 b_2}{\partial x \partial u}$$

$$-\frac{\partial c_1}{\partial x}u_z + \frac{\partial d_1}{\partial x} - k\ u_x^2\ \frac{\partial^2 b_2}{\partial u^2} + u_x\left[-\frac{\partial c_1}{\partial u}u_z + \frac{\partial d_1}{\partial u}\right]$$

$$-k\frac{\partial^2 b_2}{\partial z \partial u}\ u_z + \frac{\partial c_1}{\partial z}u_x + \frac{\partial c_2}{\partial z} - k\ \frac{\partial^2 b_2}{\partial u^2}\ u_z^2 + u_z\left[\frac{\partial c_1}{\partial u}u_x + \frac{\partial c_2}{\partial u}\right] = 0.$$

Splitting the above equation, we obtain

$$u_x^2 : -k\frac{\partial^2 b_2}{\partial u^2} = 0.$$

$$u_z^2 : -k\frac{\partial^2 b_2}{\partial u^2} = 0 \tag{3.20}$$

$$u_x : -\frac{\partial a_2}{\partial t} - k\frac{\partial^2 b_2}{\partial x \partial u} + \frac{\partial d_1}{\partial u} + \frac{\partial c_1}{\partial z} = 0$$

$$u_z : \frac{\partial b_1}{\partial u} + l(u)\frac{\partial b_2}{\partial u} - \frac{\partial c_1}{\partial x} - k\frac{\partial^2 b_2}{\partial z \partial u} + \frac{\partial c_2}{\partial u} = 0$$

$$\text{rest}: \frac{\partial b_2}{\partial t} + p(u)\frac{\partial b_2}{\partial u} + \frac{\partial d_1}{\partial x} + \frac{\partial c_2}{\partial z} = 0$$

From equation (3.20), we have

$$b_2 = A_1(t, x, z)u + A_2(t, x, z).$$

Substituting this value of b_2 in the above three equations, we obtain

$$-\frac{\partial a_2}{\partial t} - k\frac{\partial A_1}{\partial x} + \frac{\partial d_1}{\partial u} + \frac{\partial c_1}{\partial z} = 0 \tag{3.21}$$

$$\frac{\partial b_1}{\partial t} + l(u)A_1 - \frac{\partial c_1}{\partial x} - k\frac{\partial A_1}{\partial z} + \frac{\partial c_2}{\partial u} = 0 \tag{3.22}$$

$$\frac{\partial A_1}{\partial t}u + \frac{\partial A_2}{\partial t} + p(u)A_1 + \frac{\partial d_1}{\partial x} + \frac{\partial c_2}{\partial z} = 0 \tag{3.23}$$

From equation (3.21), we obtain

$$\frac{\partial d_1}{\partial u} = \frac{\partial a_2}{\partial t} + k\frac{\partial A_1}{\partial x} - \frac{\partial c_1}{\partial z},$$

and differentiating with respect to x, we obtain

$$\frac{\partial^2 d_1}{\partial x \partial u} = \frac{\partial^2 a_2}{\partial x \partial t} + k\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 c_1}{\partial x \partial z} \tag{3.24}$$

We now differentiate equation (3.23) with respect to u to obtain

$$\frac{\partial^2 d_1}{\partial u \partial x} = -\frac{\partial A}{\partial t} - p'(u)\, A_1 - \frac{\partial^2 c_2}{\partial u \partial z}. \tag{3.25}$$

Now equations (3.24) and (3.25) give us

$$\frac{\partial^2 a_2}{\partial x \partial t} + k\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 c_1}{\partial x \partial z} + \frac{\partial A_1}{\partial t} + p'(u)A_1 + \frac{\partial^2 c_2}{\partial u \partial z} = 0. \tag{3.26}$$

Differentiating equation (3.22) with respect to z, we obtain

$$\frac{\partial^2 b_1}{\partial z \partial t} + l(u)\frac{\partial A_1}{\partial z} - \frac{\partial^2 c_1}{\partial x \partial z} - k\frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 c_2}{\partial z \partial u} = 0. \tag{3.27}$$

From equation (3.17), we have

$$\frac{\partial^2 a_2}{\partial t \partial x} = \frac{\partial^2 b_1}{\partial t \partial z}. \tag{3.28}$$

Substracting equation (3.27) from (3.26), and using equation (3.28) we obtain

$$k\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial A_1}{\partial t} + p'A_1 - l\frac{\partial A_1}{\partial z} + k\frac{\partial^2 A_1}{\partial z^2} = 0,$$

that is

$$k(u)\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial z^2}\right)-l(u)\frac{\partial A_1}{\partial z}+p'(u)A_1+\frac{\partial A_1}{\partial t}=0. \tag{3.29}$$

Rewriting our conserved vector T^1, T^2 and T^3 again, we obtain

$$T^1=-a_2(t,x,z)u_x+b_1(t,x,z)u_z+A_1(t,x,z)u+A_2(t,x,z),$$

$$T^2=a_2(t,x,z)u_t-k(u)u_xA_1(t,x,z)-c_1(t,x,z,u)u_z+d_1(t,x,z,u),$$

$$T^3=-b_1(t,x,z)u_t-k(u)u_zA_1(t,x,z)+c_1(t,x,z,u)u_x+c_2(t,x,z,u),$$

and relabelling $a_2\longrightarrow a$, $b_1\longrightarrow b$, $A_1\longrightarrow A$, $A_2\longrightarrow B$, $d_1\longrightarrow d$, gives us the conserved vector

$$\begin{aligned}
T^1&=-a(t,x,z)u_x+b(t,x,z)u_z+A(t,x,z)u+B(t,x,z),\\
T^2&=a(t,x,z)u_t-k(u)u_xA(t,x,z)-c_1(t,x,z,u)u_z+d(t,x,z,u),\\
T^3&=-b(t,x,z)u_t-k(u)u_zA(t,x,z)+c_1(t,x,z,u)u_x+c_2(t,x,z,u),
\end{aligned} \tag{3.30}$$

where the functions a to B satisfy the following system of equations:

$$\begin{aligned}
\frac{\partial a}{\partial x}-\frac{\partial b}{\partial z}&=0,\\
\frac{\partial a}{\partial t}+k(u)\frac{\partial A}{\partial x}-\frac{\partial d}{\partial u}-\frac{\partial c_1}{\partial z}&=0,\\
\frac{\partial b}{\partial t}+l(u)A-\frac{\partial c_1}{\partial x}-k(u)\frac{\partial A}{\partial z}+\frac{\partial c_2}{\partial u}&=0,\\
\frac{\partial A}{\partial t}u+\frac{\partial B}{\partial t}+p(u)A+\frac{\partial d}{\partial x}+\frac{\partial c_2}{\partial z}&=0.
\end{aligned} \tag{3.31}$$

Also, equation (3.29) becomes

$$k(u)\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial z^2}\right)-l(u)\frac{\partial A}{\partial z}+p'(u)A+\frac{\partial A}{\partial t}=0. \tag{3.32}$$

The use of equation (3.32) enables us to classify all cases for which conservation laws for equation (3.2) exist.

We first consider the case when the functions $k(u), l(u)$ and $p(u)$ are arbitrary. From equation (3.32), we obtain $A=0$. Thus

$$T^1=-au_x+bu_z+B,$$

$$T^2 = au_t - c_1u_z + d,$$

$$T^3 = -bu_t + c_1u_x + c_2.$$

Now

$$D_tT^1 + D_xT^2 + D_zT^3 = -a_tu_x - au_{xt} + b_tu_z + bu_{zt} + B_t + a_xu_t + au_{tx}$$

$$-c_{1x}u_z - c_1u_{zx} - c_{1u}u_xu_z + d_x + d_uu_x - b_zu_t - bu_{tz} + c_{1z}u_x + c_1u_{xz} + c_{2z} + c_{2u}u_z + c_{1u}u_zu_x$$

$$= -a_tu_x + b_tu_z + B_t - c_{1x}u_z + d_x + d_uu_x + c_{1z}u_x + c_z + c_uu_z$$

or

$$D_tT^1 + D_xT^2 + D_zT^3 = (-a_t + d_u + c_{1z})u_x + (b_t - c_{1x} + c_{zu})u_z + B_t + d_x + c_{2z}$$

and so

$$D_tT^1 + D_xT^2 + D_zT^3 = (kA_x)u_x + (-lA + kA_z)u_z - A_tu - pA = 0,$$

because $A = 0$. Thus $D_tT^1 + D_xT^2 + D_zT^3 = 0$ is satisfied without the equation, which means that the conservation law is trivial.

We now consider other cases. Differentiating the equation

$$k(u)\left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial z^2}\right) - l(u)\frac{\partial A}{\partial z} + p'(u)A + \frac{\partial A}{\partial t} = 0$$

with respect to u, we obtain

$$k'(A_{xx} + A_{zz}) - l'A_z + p''A = 0. \tag{3.33}$$

CASE A

If $p'' = 0$ (that is, $p(u) = p_0 + p_1(u)$, where p_0 and p_1 are constants), then we have

$$k'(A_{xx} + A_{zz}) - l'A_z = 0.$$

CASE A(i)

If $l' = 0$ (that is, $l = l_0$, where l_0 is a constant), then $k'(A_{xx} + A_{zz}) = 0$.

CASE A(i)1

If $k' = 0$ (that is, $k = k_0$, where k_0 is a constant), then we have

$$k_0(A_{xx} + A_{zz}) - l_0A_z + p_1A + A_t = 0.$$

CASE A(i)2

If $k' \neq 0$ (that is, $k = k(u)$), then we have $A_{xx} + A_{zz} = 0$ and $-l_0 A_z + p_1 A + A_t = 0$.

CASE A(ii)

If $l' \neq 0$, (that is, $l = l(u)$), from equation (3.33), we obtain

$$\frac{k'}{l'}\,(A_{xx} + A_{zz}) - A_z = 0.$$

Differentiating this equation with respect to u, we obtain

$$\left(\frac{k'}{l'}\right)'\,(A_{xx} + A_{zz}) = 0. \tag{3.34}$$

CASE A(ii)1

If $\left(\frac{k'}{l'}\right)' = 0$ (that is, $k' = l'N$ or $k = Nl + M$, where N and M are constants), we have from equation (3.34),

$$N(A_{xx} + A_{zz}) - A_z = 0 \tag{3.35}$$

and from equation (3.32), we obtain

$$(Nl + M)(A_{xx} + A_{zz}) - l(u)A_z + p_1 A + A_t = 0$$

which gives us

$$M(A_{xx} + A_{zz}) + p_1 A + A_t = 0.$$

CASE A(ii)2

If $\left(\frac{k'}{l'}\right)' \neq 0$ (that is $k \neq Nl + M$), we have

$$A_{xx} + A_{zz} = 0$$

and so from equation (3.35), we obtain $A_z = 0$. This gives $A_{xx} = 0$ and so $p_1 A + A_t = 0$.

CASE B

If $p'' \neq 0$, from equation (3.33), we have

$$\frac{k'}{p''}(A_{xx} + A_{zz}) - \frac{l'}{p''}A_z + A = 0. \tag{3.36}$$

Differentiating the above equation with respect to u, we obtain

$$\left(\frac{k'}{p''}\right)'(A_{xx}+A_{zz})-\left(\frac{l'}{p''}\right)'A_z=0. \tag{3.37}$$

CASE B(i)

If $\left(\frac{l'}{p''}\right)'=0$, (that is, $\frac{l'}{p''}=N$, or $l=Np'+M$) we obtain

$$\left(\frac{k'}{p''}\right)'(A_{xx}+A_{zz})=0. \tag{3.38}$$

CASE B(i)1

If $\left(\frac{k'}{p''}\right)'=0$ (that is, $k'=Rp''$, or $k=Rp'+S$, where R and S are constants), from equation (3.36), we obtain

$$R(A_{xx}+A_{zz})-NA_z+A=0 \tag{3.39}$$

and from equation (3.32), we obtain

$$(Rp'+S)(A_{zz}+A_{xx})-(Np'+M)A_z+p'A+A_t=0$$

and so we have

$$S(A_{xx}+A_{zz})-MA_z+A_t=0. \tag{3.40}$$

CASE B(i)2

If $\left(\frac{k'}{p''}\right)'\neq 0$, then from equation (3.38), we obtain $A_{xx}+A_{zz}=0$ and equation (3.36) gives

$$-\frac{l'}{p''}A_z+A=0.$$

Since $l'=Np''$, we obtain

$$-NA_z+A=0 \tag{3.41}$$

and equation (3.32) gives

$$-(Np'+M)A_z+p'A+A_t=0. \tag{3.42}$$

Using equation (3.41), we obtain

$$-MA_z+A_t=0. \tag{3.43}$$

CASE B(ii)

If $\left(\frac{l'}{p''}\right)' \neq 0$, we have from equation (3.37),

$$\frac{\left(\frac{k'}{p''}\right)'}{\left(\frac{l'}{p''}\right)'}(A_{zz} + A_{xx}) - A_z = 0. \tag{3.44}$$

Differentiating with respect to u, we obtain

$$\left[\frac{\left(\frac{k'}{p''}\right)'}{\left(\frac{l'}{p''}\right)'}\right]'(A_{zz} + A_{xx}) = 0.$$

CASE B (ii)1

If

$$\left[\frac{\left(\frac{k'}{p''}\right)'}{\left(\frac{l'}{p''}\right)'}\right]' = 0$$

that is, if

$$\frac{\left(\frac{k'}{p''}\right)'}{\left(\frac{l'}{p''}\right)'} = L,$$

or

$$\frac{k'}{p''} = L\left(\frac{l'}{p''}\right) + Q$$

where L and Q are constants, from equations (3.44) and (3.36), we obtain the following equations:

$$L(A_{xx} + A_{zz}) - A_z = 0, \tag{3.45}$$

$$\left[L\left(\frac{l'}{p''}\right) + Q\right](A_{zz} + A_{xx}) - \frac{l'}{p''}A_z + A = 0. \tag{3.46}$$

Now using equation (3.44) in equation (3.45), we obtain

$$Q(A_{zz} + A_{xx}) + A = 0. \tag{3.47}$$

Also since $k' = Ll' + p''Q$, we have $k(u) = Ll + p'Q + R$, where R is a constant. Thus equation (3.32) becomes

$$(Ll + p'Q + R)(A_{zz} + A_{xx}) - lA_z + p'A + A_t = 0$$

and using equations (3.44) and (3.46), we obtain

$$R(A_{xx} + A_{zz}) + A_t = 0. \tag{3.48}$$

CASE B(ii)2

If

$$\left[\frac{\left(\frac{k'}{p''}\right)}{\left(\frac{l'}{p''}\right)}\right]' \neq 0,$$

then

$$A_{zz} + A_{xx} = 0$$

and so from equation (3.43), we have $A_z = 0$, and from equation (3.36), we obtain $A = 0$ and so we have only trivial conservation laws for this case.

We now summarize the above results:

1. For arbitrary $k(u), l(u)$ and $p(u)$ we obtain $A = 0$ and we get trivial conservation laws.

Nontrivial conservation laws are obtained in the following cases:

2. $p = p_0 + p_1 u, l = l_0, k = k_0 \neq 0$, where p_0, p_1, l_0 and k_0 are arbitrary constants. The function A satisfies

$$(A_{xx} + A_{zz})k_0 - l_0 A_z + p_1 A + A_t = 0$$

together with the system (3.31).

3. $p = p_0 + p_1 u, l = l_0, k = k(u)$ with $k(u) \neq 0$, where p_0, p_1 and l_0 are arbitrary constants and A satisfies

$$-l_0 A_z + p_1 A + A_t = 0$$

$$A_{xx} + A_{zz} = 0$$

We solve for A as follows:

$$\frac{dx}{0} = \frac{dz}{-l_0} = \frac{dt}{1} = \frac{dA}{-p_1 A}$$

This gives

$$z + l_0 t = c_1, \ x = c_3$$

$$lnA + p_1 t = lnc - 2$$

or

$$\frac{A}{c_2} = \exp(-p_1 t)$$

$$i.e\ A = c_2 \exp(-p_1 t)$$

$$c_2 = f(c_1, c_3)$$

Thus

$$A = \exp(-p_1 t) f(x, z + l_0 t),$$

and

$$A_{xx} + A_{zz} = 0$$

Implies that f must satisfy

$$f_{xx} + f_{\tau\tau} = 0$$

where $\tau = z + l_0 t$. The system (3.31) also must be satisfied.

4. $p = p_0 + p_1 u$, $l = l(u)$ with $l'(u) \neq 0, k = Nl + M$, where p_0, p_1, N and M are arbitrary constants. In this case A satisfies the two equations

$$N(A_{zz} + A_{xx}) - A_z = 0 \tag{3.49}$$

and

$$M(A_{zz} + A_{xx}) + p_1 A + A_t = 0 \tag{3.50}$$

From (4.36) we have

$$A_{zz} + A_{xx} = \frac{A_z}{N}$$

and substituting this value of $A_{xx} + A_{zz}$ in (4.37), we obtain

$$\frac{M}{N} A_z + p_1 A + A_t = 0$$

or

$$M A_z + N A_t = -p_1 N A$$

We have

$$\frac{dz}{M} = \frac{dt}{N} = \frac{dA}{-p_1 N A} = \frac{dx}{0}$$

which gives $Nz - Mt = c_1$, $Ae^{p_1 t} = c_2$ and $x = c_3$ and finally, we obtain

$$A = e^{-p_1 t} f(x, Nz - Mt),$$

Substituting this value of A in equation (3.36), we obtain

$$N(e^{-p_1 t} f_{xx} + e^{-p_1 t} f_{\tau\tau} N^2)) - e^{-p_1 t} f_\tau N = 0$$

or

$$f_{xx} + N^2 f_{\tau\tau} - f_\tau = 0$$

Thus

$$A = \exp(-p_1 t) f(x, Nz - Mt)$$

where f satisfies $f_{xx} + N^2 f_{\tau\tau} - f_\tau = 0$, $\tau = Nz - Mt$. The system (3.31) needs to be satisfied as well.

5. $p = p_0 + p_1 u, l = l(u)$ with $l'(u) \neq 0, k = k(u)$ with $k(u) \neq Nl(u) + M$, where p_0, p_1, N and M are arbitrary constants, and A satisfies $A_z = 0$, $A_{xx} = 0$ and $p_1 A + A_t = 0$. From the first two equations, we have

$$A \equiv A(t, x) = x f(t) + g(t)$$

and substituting in the third equation, we obtain

$$p_1(x f(t) + g(t)) + f'(t) + g'(t) = 0$$

Splitting on 'x', we have $p_1 f + f' = 0$ and $p_1 g + g' = 0$ and this gives $f(t) = f_0 \exp(-p_1 t)$ and $g(t) = g_0 \exp(-p_1 t)$, whose f_0 and g_0 are arbitrary constants.Thus in this case

$$A = f_0 x \exp(-p_1 t) + g_0 \exp(-p_1 t)$$

and it must satisfy the system (3.31).

6. $p = p(u)$ with $p''(u) \neq 0, l(u) = Np'(u) + M$, and $k(u) = Rp'(u) + S$, where N, M, R and S are arbitrary constants. Here the function A satisfies the following two equations:

$$R(A_{xx} + A_{zz}) - NA_z + A = 0 \tag{3.51}$$

and

$$S(A_{xx}+A_{zz})-MA_z+A_t=0 \tag{3.52}$$

From equation (3.39) we have

$$A_{xx}+A_{zz}=\frac{N}{R}A_z-\frac{A}{R}, \text{ provided } R\neq 0$$

Substituting this value of $A_{xx}+A_{zz}$ in equation (3.40), we have

$$S\left(\frac{N}{R}A_z-\frac{A}{R}\right)-MA_z+A_t=0$$

or

$$(SN-MR)\frac{\partial A}{\partial z}+R\frac{\partial A}{\partial t}=SA$$

Thus we have

$$\frac{dz}{SN-MR}=\frac{dt}{R}=\frac{dA}{SA}=\frac{dx}{0}$$

and this gives

$$c_1=(SN-MR)t-Rz, \quad c_2=A\exp(-\frac{S}{R}t) \text{ and } c_3=x$$

Hence

$$A=\exp(\frac{S}{R}t)f(x,(SN-MR)t-Rz,R\neq 0) \tag{3.53}$$

and substituting this value of A in equation (3.38), we obtain

$$R\{e^{\frac{St}{R}}f_{xx}+e^{\frac{St}{R}}f_{\tau\tau}R^2\}+Ne^{\frac{St}{R}}f_\tau R+e^{\frac{St}{R}}f=0$$

Where

$$\tau=(SN-MR)t-Rz$$

Thus A is given by equation (3.40), where f solve

$$Rf_{xx}+R^3f_{\tau\tau}+NRf_\tau+f=0, \quad \tau=(SN-MR)t-Rz$$

Further A is constrained by the system (3.31). Now if $R=0$, from equation (3.39), we obtain

$$-NA_z+A=0$$

which can be solved for A to obtain

$$A = \exp(\frac{z}{N})f(t,x), \quad N \neq 0, \quad R = 0 \tag{3.54}$$

Substituting this value of A in equation (3.40), we obtain

$$S\left(\exp\left(\frac{z}{N}\right)f_{xx} + \exp\left(\frac{z}{N}\right)\frac{1}{N^2}f\right) - \frac{M}{N}\exp\left(\frac{z}{N}\right)f + \exp\left(\frac{z}{N}\right)f_t = 0.$$

Thus if $R = 0$, the function A is given by equation (3.54), where f satisfies

$$SN^2 f_{xx} + (s - MN)f + N^2 f_t = 0$$

and f is further constrained by the system (3.31).

7. $p = p(u)$ with $p''(u) \neq 0, l(u) = Np'(u) + M, k = k(u)$ with $k(u) \neq Rp'(u) + S$, where N, M, R and S are arbitrary constants. In this case A satisfies the following three conditions:

$-NA_z + A = 0, \quad -MA_z + A_t = 0$ and $A_{xx} + A_{zz} = 0$. Solving the first equation gives

$$A = \exp\left(\frac{z}{N}\right)f(t,x)$$

and substituting the second equation yields a first order partial differential equation, which on solving gives

$$f(t,x) = \exp\left(\frac{Mt}{N}\right)g(x)$$

where $g(x)$ is an arbitrary function. Thus

$$A = \exp\left(\frac{z + Mt}{N}\right)g(x)$$

and substituting this value of A in the third equation yields

$$g'' + \frac{1}{N^2}\,g = 0$$

which gives

$$g(x) = f_0 \cos\left(\frac{x}{N}\right) + g_0 \sin\left(\frac{x}{N}\right), \quad N \neq 0$$

where f_0 and g_0 are constants. Thus

$$A = \exp\left(\frac{z + Mt}{N}\right)\left[f_0 \cos\left(\frac{x}{N}\right) + g_0 \sin\left(\frac{x}{N}\right)\right]$$

and it must also satisfy the system (3.31).

8. $p = p(u)$ with $p''(u) \neq 0, l = l(u)$ with $l(u) \neq Np'(u) + M$, $k(u) = Ll(u) + Qp'(u) + R$, where L, Q, R, M and N are arbitrary constants. Here A satisfies the three equations:

$$L(A_{xx} + A_{zz}) - A_z = 0 \tag{3.55}$$

$$Q(A_{xx} + A_{zz}) + A = 0 \tag{3.56}$$

and

$$R(A_{xx} + A_{zz}) + A_t = 0 \tag{3.57}$$

Substituting the value of $A_{xx} + A_{zz}$ from equation (3.47) into equations (3.48) and (3.45), we obtain, respectively

$$-\frac{R}{Q}A + A_t = 0$$

and

$$-\frac{L}{Q}A - A_z = 0.$$

Solving the first equation gives

$$A = \exp\left(\frac{Rt}{Q}\right) f(x, z)$$

and substituting this value of A in the second equation, we obtain

$$-\frac{L}{Q}f - f_z = 0.$$

The solution of this equation is

$$f = \exp\left(-\frac{L}{Q}\right) zg(x)$$

and hence

$$A = \exp\left(\frac{Rt - Lz}{Q}\right) g(x).$$

Now substituting this value of A in equation (3.48) gives :

$$g_{xx} + \frac{L^2}{Q^2}\, g + \frac{1}{Q}g = 0$$

Hence the above yields

$$\begin{aligned}
T^1 &= f(x, z + l_0 t) u \exp(-p_1 t), \\
T^2 &= -k(u) u_x \exp(-p_1 t) f(x, z + l_0 t) + \exp(-p_1 t) f_x \int_0^u k(u') du' + \alpha(t, x, z), \\
T^3 &= -k(u) u_z \exp(-p_1 t) f(x, z + l_0 t) + \exp(-p_1 t) f_\tau \int_0^u k(u') du' \\
&\quad - l_0 u f \exp(-p_1 t) + \beta(t, x, z), \qquad (3.64)
\end{aligned}$$

where f satisfies $f_{xx} + f_{\tau\tau} = 0$, $\tau = z + l_0 t$ and α and β are constrained by $p_0 \exp(-p_1 t) f + \alpha_x + \beta_z = 0$. Equations (3.64) give rise to an infinite number of conserved vectors.

[Verification:

$$\begin{aligned}
& D_t T^1 + D_x T^2 + D_z T^3 = -p_1 u \exp(p_t) f_x + \exp(-p_1 t) f_{xx} \int_0^u k(u') du' + \alpha_x \\
& + u_x\{-k' u_x \exp(-p_1 t) f + \exp(-p_1 t) f_x k\} - u_{xx} k \exp(-p_1 t) f - k u_z \exp(-p_1 t) f_\tau \\
& + \exp(-p_1 t) f_{\tau\tau} \int_0^u k(u') du' - l_0 u \exp(-p_1 t) f_\tau + \beta_z \\
& + u_z\{-k' u_z \exp(-p_1 t) f + \exp(-p_1 t) f_\tau k - l_0 \exp(-p_1 t) f\} - u_{zz} k \exp(-p_1 t) f \\
& = \exp(-p_1 t) f\{-p_1 u + u_t - k' u_x^2 - k u_{xx} - k' u_z^2 - l_0 u_z - k u_{zz}\} + u l_0 \exp(-p_1 t) f \tau \\
& - k u_x \exp(-p_1 t) f_x + k u_x \exp(-p_1 t) f_x - k u_z \exp(-p_1 t) f_\tau - l_0 u \exp(-p_1 t) f_\tau \\
& + k u_z \exp(-p_1 t) f \tau + \alpha_x + \beta_z \\
& = p_0 \exp(-p_1 t) f + \alpha_x + \beta_z = 0.]
\end{aligned}$$

Similarly, for Case 5 with the choices $a = b = c_1 = B = 0$, we obtain

$$\begin{aligned}
T^1 &= [f_0 x \exp(-p_1 t) + g_0 \exp(-p_1 t)] u, \\
T^2 &= -k(u) u_x [f_0 x \exp(-p_1 t) + g_0 \exp(-p_1 t)] + f_0 \exp(-p_1 t) \int_0^u k(u') du' + \alpha(t, x, z), \\
T^3 &= -k(u) u_z [f_0 x \exp(-p_1 t) + g_0 \exp(-p_1 t)] - [f_0 x \exp(-p_1 t) \\
&\quad + g_0 \exp(-p_1 t)] \int_0^u l(u') du' + \beta(t, x, z), \qquad (3.65)
\end{aligned}$$

where α and β are constrained by

$$p_0 [f_0 x \exp(-p_1 t) + g_0 \exp(-p_1 t)] + \alpha_x + \beta_z = 0.$$

[Verification:

$$D_tT^1 + D_xT^2 + D_zT^3 = -p_1\exp(-p_1t)[f_0x+g_0]u + u_t\exp(-p_1t)[f_0x+g_0]$$
$$-ku_x\exp(-p_1t)f_0 + u_x[-k'u_x\exp(-p_1t)[f_0x+g_0] + f_0\exp(-p_1t)k]$$
$$+u_{xx}[-k\exp(-p_1t)[f_0x+g_0]] + \alpha_x + \beta_z + u_z[-k'u_z\exp(-p_1t)[f_0x+g_0]]$$
$$= \exp(-p_1t)[f_0+g_0]\{-p_1u + u_t - k'u_x^2 - ku_{xx} - k'u_z^2 - lu_z - ku_{zz}\} + \alpha_x + \beta_z$$
$$= \exp(-p_1t)[f_0x+g_0]\{p_0\} + \alpha_x + \beta_z = 0.]$$

Thus the components (3.65) result in two conserved vectors.

The conservation laws for the other cases can be constructed in a similar fashion.

The only classes in the symmetry classification (see [4]) which have nontrivial conservation laws are (the notation used in the following corresponds to that of [4])

I.2. $k(u)$ arbitrary, $p(u) = 0, l(u) = 0$.

II. $k(u) = e^u$

1. $l(u) = Ae^u, p(u) = Be^u + C$ (A, B and C are arbitrary constants, $A \neq 0$)
4. $l(u) = 0$, (i) $p = \pm e^u + \delta, \delta = \pm 1$, (iii) $p(u) = \delta, \delta \pm 1$, (iv) $p(u) = 0$.

III. $k(u) = u^\sigma, \sigma \neq 0, -1$

1. $l(u) = Au^\sigma, p(u) = Bu^{\sigma+1} - (C/\sigma)u$ (A, B, C and σ are constants, $A \neq 0$)
2. $l(u) = Cu^\mu, p(u) = Au^{1+2\mu-\sigma}, \mu = 2\sigma$
3. $l(u) = 0$ (i) $p(u) = \pm u^\nu, \nu \neq 0, 1, \nu = \sigma+1$, (ii) $p(u) = \pm u^{\sigma+1} + \delta u, \delta = \pm 1$, $\sigma = \text{const}$ (iii) $p(u) = \delta u, \delta \pm 1$, (iv) $p(u) = 0$.

IV. $k(u) = u^{-1}, l(u) = 0$

(ii) $p(u) = \delta u \pm 1, \delta = \pm 1$ (iii) $p(u) = \pm 1$, (iv) $p(u) = \delta u, \delta = \pm 1$,
(v) $p(u) = 0$

V. $k(u) = 1$,

3. $l(u) = A\ln u, p(u) = u(B\ln u + C)$
6. $l(u) = Au, p(u) = Bu + C$
8. $l(u) = 0$ (i) $p(u) = \delta u, \delta = \pm 1$, (ii) $p(u) = \pm 1$, (vi) $p(u) = 0$.

Chapter 4

Conclusion

In this project we have first reviewed some useful definitions and theorems of modern group analysis which were later used in our work. We have determined exact/asymptotic invariant solutions of certain soil water equations using reduction by Lie symmetry subalgebras. Furthermore, we have obtained all non-trivial conservation laws for soil water type equations. It has been shown that for arbitrary functions these equations possess trivial conservation laws. Seven cases arise for which we have nontrivial conserved vectors. Among these, three cases result in each admitting two nontrival conserved vectors. Each of the other cases gives us infinite number of nontrivial conservation laws. We also provided all the classes in the symmetry classification which have nontrivial conserved vectors.

Further work can be done on the reduction and solutions of the soil water equations which admit symmetries that preserve the conservation laws.

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