LIE GROUP ANALYSIS OF CERTAIN NONLINEAR DIFFERENTIAL EQUATIONS ARISING IN FLUID MECHANICS

by

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Declaration

I declare that the dissertation for the degree of Master of Science at North-West University, Mafikeng Campus, hereby submitted, has not previously been submitted by me for a degree at this or any other university, that this is my own work in design and execution and that all material contained herein has been duly acknowledged.

BELINDA THEMBISA MATEBESE

15 November 2010

Dedication

To my late grandmother Daki, my family Audrey, sis Xoliswa, Mbita, Za, my father Tati and Katlego.

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Abstract

This research studies two nonlinear differential equations arising in fluid mechanics. Firstly, the Zakharov-Kuznetsov's equation in (3+1) dimensions with an arbitrary power law nonlinearity is considered. The method of Lie symmetry analysis is used to carry out the integration of Zakharov-Kuznetsov's equation. Also, the extended tanh-function method and the G'/G method are used to integrate the Zakharov-Kuznetsov's equation. The non-topological soliton solution is obtained by the aid of solitary wave ansatz method. Numerical simulation is given to support the analytical development.

Secondly, the nonlinear flow problem of an incompressible viscous fluid is considered. The fluid is taken in a channel having two weakly permeable moving porous walls. An incompressible fluid fills the porous space inside the channel. The fluid is magnetohydrodynamic in the presence of a time-dependent magnetic field. Lie group method is applied along with perturbation method in the derivation of analytic solution. The effects of the magnetic field, porous medium, permeation Reynolds number and wall dilation rate on the axial velocity are shown and discussed.

Introduction

Fluid Mechanics is one of the most important areas of study in Applied Mathematics and Theoretical Physics. This area of study and research appears in everyday lives. The study of fluid flow has a variety of applications in various scientific and engineering fields, such as aerodynamics, hydrodynamics, convection heat transfer, oceanography, dynamics of multi-phase flows etc.

Most scientific problems and phenomena that arise in fluids are modelled by nonlinear ordinary or partial differential equations. These equations are widely used to describe complex phenomena in various fields of sciences which combine different types of differential equations (see for example [1]-[5]).

There are a number of approaches for solving nonlinear partial differential equations, which range from completely analytical to completely numerical ones.

Lie group analysis, based on symmetry and invariance principles, is a systematic method for solving nonlinear differential equations analytically. Originally developed by Sophus Lie (1842-1899), the philosophy of Lie groups has become an essential part of the mathematical culture for anyone investigating mathematical models of physical, engineering and natural problems. Lie group analysis embodies and synthesizes symmetries of differential equations. A symmetry is described roughly as a change or, a transformation, that leaves an object apparently unchanged. Symmetries permeate many mathematical models, in particular those formulated in terms of differential equations.

The Zakharov-Kuznestsov (ZK) equation is a nonlinear evolution equation (NLEE) that has been studied for the past decades. The equation was first derived for describing weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [6]. The ZK equation

$$u_t + auu_x + b\left(u_{xx} + u_{yy}\right)_x = 0$$

and

$$u_t + auu_x + b\left(u_{xx} + u_{yy} + u_{zz}\right)_x = 0$$

is the best known two- and three-dimensional generalization of the KdV equation investigated in [7, 8, 9]. The ZK equation is not integrable by the inverse scattering transform method. It was found that the solitary-wave solutions of the ZK equation are inelastic [10]. Shivamoggi [11] showed that it possesses the Painlevè property by making a Painlevè analysis of the ZK equation.

Several researchers have used various methods, for example Exp-function method [12], Homotopy perturbation method and variational iteration method [13], among others, to solve the ZK equation in (2+1)-dimensions. Biswas and Zerrad [14] and [15] considered the ZK equation with dual-power law nonlinearity and later considered the ZK equation in plasmas with power law nonlinearity to obtain 1-soliton solution using the solitary wave ansatz method. Biswas [16] used the solitary wave ansatz method to obtain 1-soliton solution of the generalized ZK equation with nonlinear dispersion and time-dependent coefficient. Deng [17] applied extended hyperbolic function method to (2+1)-dimensional Zakharov-Kuznetsov (ZK) equation and its generalized form and obtained new explicit and exact solitary wave, multiple nontrivial exact periodic travelling wave solutions, solitons solutions and complex solutions. Wazwaz [18] employed the sine-cosine ansatz and obtained exact solutions with soliton and periodic structures for the (2+1) and (3+1) ZK equation and its modified form. Later Wazwaz [10] used the extended tanh method to study the ZK equation, the generalized ZK equation, the modified ZK equation and a generalized form of the modified ZK equation and obtained new solitons and periodic solutions.

In this research we study the ZK equation with power law nonlinearity in (3+1) dimensions given by

$$q_t + aq^n q_x + b\left(q_{xx} + q_{yy} + q_{zz}\right)_x = 0, \tag{1}$$

where a, b and n are constants. In equation (1), the first term represents the evolution term while a represents the coefficients of power law nonlinearity, and b is the coefficient of dispersion terms. The parameter n is the power law parameter while q is the wave profile. The independent variables x, y, z and t represent spatial and temporal variables respectively.

In the second part of our research we consider two-dimensional flow in a deformable channel with porous medium and variable magnetic field.

The two-dimensional flow of viscous fluid in a porous channel appears very useful in many applications. Hence many experimental and theoretical attempts have been made in the past. Such studies have been presented under the various assumptions like small Reynolds number (R_e), intermediate R_e, large R_e and arbitrary R_e. The steady flow in a channel with stationary walls and small R_e has been studied by Berman [19]. Dauenhaver and Majdalani [20] numerically discussed the two-dimensional viscous flow in a deformable channel when $-50 < R_e < 200$ and $-100 < \alpha < 100$ (α denotes the wall expansion ratio). In another study, Majdalani et al. [21] analyzed the channel flow of slowly expanding-contracting walls which leads to the transport of biological fluids. They first derived the analytic solution for small R and α and then compared it with the numerical solution. The flow problem given in study [21] has been analytically solved by Boutros et al. [22] when Reynolds number and α vary in the ranges $-5 < R_e < 5$ and $-1 < \alpha < 1$. They used the Lie group method in this study. Mahmood et al. [23] discussed the homotopy perturbation and numerical solutions for viscous flow in a deformable channel with porous medium. Asghar et al. [24] computed exact solution for the flow of viscous fluid through expanding-contracting channels. They used symmetry method and conservation laws.

In this research work we generalize the flow analysis of [22] with the influence of magnetic field and porous medium given by

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{2}$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial \bar{x}} + s\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right] - \frac{s\phi}{k}\bar{u} - \frac{r\delta^2 H^2(t)}{\rho}\bar{u}, \tag{3}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial \bar{y}} + s\left[\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right] - \frac{s\phi}{k}\bar{v}, \tag{4}$$

with the following boundary conditions

$$\begin{aligned}
&\text{(i)} \quad && \bar{u} = 0, \quad \bar{v} = -V_w = -A\dot{a} \quad \text{at } \bar{y} = a(t), \\
&\text{(ii)} \quad && \frac{\partial \bar{u}}{\partial \bar{y}} = 0, \quad \bar{v} = 0 \quad \text{at } \bar{y} = 0, \\
&\text{(iii)} \quad && \bar{u} = 0 \quad \text{at } \bar{x} = 0,
\end{aligned} \tag{5}$$

where $\cdot$ denotes the differentiation with respect to t. In the above expressions $\bar{u}$ and $\bar{v}$ are the velocity components in $\bar{x}$ (axial coordinate) and $\bar{y}$ (normal coordinate) directions, respectively, ρ is the fluid density, $\bar{P}$ is the pressure, t is the time, δ is the kinematic viscosity, ϕ and k are the porosity and permeability of porous medium, respectively, r is the electrical conductivity of fluid, V_w is the fluid inflow velocity, A is the injection coefficient corresponding to the porosity of wall and $\phi = V_f/V_c$, where V_f and V_c, respectively, indicate the volume of the fluid and control volume.

The outline of the research project is as follows:

In Chapter 1 the basic definitions and theorems concerning the Lie group method are recalled. Chapter 2 deals with the construction of exact solutions of the ZK equation with power law nonlinearity in (3+1) dimensions using the Lie group method, extended tanh method, (G'/G)-expansion method and solitary wave ansatz method. In Chapter 3 Lie group analysis is applied along with perturbation method to obtain an analytical solution for the nonlinear flow problem of an incompressible viscous fluid and then compare it with the numerical solution. Chapter 4 summarizes the results of the dissertation.

Bibliography is given at the end.

Chapter 1

Lie group theory of PDEs

In this chapter a brief introduction to the Lie group theory of partial differential equations is given. This includes the algorithm to determine the Lie point symmetries of partial differential equations.

1.1 Introduction

More than a hundred years ago, the Norwegian mathematician Marius Sophus Lie realized that many of the methods for solving differential equations could be unified using group theory. He developed a symmetry-based approach to obtaining exact solutions of differential equations. Symmetry methods have great power and generality - in fact, nearly all well-known techniques for solving differential equations are special cases of Lie's methods. Recently, several books have been written on this topic. We list a few of them here. Ovsiannikov [25], Olver [26], Bluman and Kumei [27], Ibragimov [28].

The definitions and results presented in this chapter are taken from the books mentioned above.

1.2 Continuous one-parameter groups

Let $x = (x^1, ..., x^n)$ be the independent variables with coordinates x^i and $q = (q^1, .., q^m)$ be the dependent variables with coordinates q^α (n and m finite). Consider a change of the variables x and q involving a real parameter a:

$$T_a : \bar{x^i} = f^i(x, q, a),\ \bar{q^\alpha} = \phi^\alpha(x, q, a), \tag{1.1}$$

where a continuously ranges in values from a neighborhood $\mathcal{D}' \subset \mathcal{D} \subset \mathbb{R}$ of $a = 0$, and f^i and ϕ^α are differentiable functions.

Definition 1.1 A set G of transformations (1.1) is called a *continuous one-parameter (local) Lie group of transformations* in the space of variables x and q if

(i) For T_a, $T_b \in G$ where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then $T_b T_a = T_c \in G$, $c = \phi(a, b) \in \mathcal{D}$ (Closure)

(ii) $T_0 \in G$ if and only if $a = 0$ such that $T_0 T_a = T_a T_0 = T_a$ (Identity)

(iii) For $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$, $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that $T_a T_{a^{-1}} = T_{a^{-1}} T_a = T_0$ (Inverse)

We note that the associativity property follows from (i). The group property (i) can be written as

$$\begin{aligned} \bar{\bar{x}}^i &\equiv f^i(\bar{x}, \bar{q}, b) = f^i(x, q, \phi(a, b)), \\ \bar{\bar{q}}^\alpha &\equiv \phi^\alpha(\bar{x}, \bar{q}, b) = \phi^\alpha(x, q, \phi(a, b)) \end{aligned} \tag{1.2}$$

and the function ϕ is called the *group composition law*. A group parameter a is called *canonical* if $\phi(a, b) = a + b$.

Theorem 1.1 For any $\phi(a, b)$, there exists the canonical parameter $\tilde{a}$ defined by

$$\tilde{a} = \int_0^a \frac{ds}{w(s)}, \text{ where } w(s) = \left.\frac{\partial \phi(s, b)}{\partial b}\right|_{b=0}.$$

1.3 Prolongation of point transformations and Group generator

The derivatives of q with respect to x are defined as

$$q_i^\alpha = D_i(q^\alpha), \quad q_{ij}^\alpha = D_j D_i(q_i), \cdots, \tag{1.3}$$

where

$$D_i = \frac{\partial}{\partial x^i} + q_i^\alpha \frac{\partial}{\partial q^\alpha} + q_{ij}^\alpha \frac{\partial}{\partial q_j^\alpha} + \cdots, \quad i = 1, ..., n \tag{1.4}$$

is the operator of total differentiation. The collection of all first derivatives q_i^α is denoted by $q_{(1)}$, i.e.,

$$q_{(1)} = \{q_i^\alpha\} \quad \alpha = 1, ..., m, \quad i = 1, ..., n.$$

Similarly

$$q_{(2)} = \{q_{ij}^\alpha\} \quad \alpha = 1, ..., m, \quad i, j = 1, ..., n$$

and $q_{(3)} = \{q_{ijk}^\alpha\}$ and likewise $q_{(4)}$ etc. Since $q_{ij}^\alpha = q_{ji}^\alpha$, $q_{(2)}$ contains only q_{ij}^α for $i \leq j$. In the same manner $q_{(3)}$ has only terms for $i \leq j \leq k$. There is natural ordering in $q_{(4)}$,$q_{(5)}$ $\cdots$.

In group analysis all variables $x, q, q_{(1)} \cdots$ are considered functionally independent variables connected only by the differential relations (1.3). Thus the q_s^α are called differential variables and a pth-order partial differential equation (PDE) is given as

$$E(x, q, q_{(1)}, ..., q_{(p)}) = 0. \tag{1.5}$$

Prolonged or extended groups

If $z = (x, q)$, one-parameter group of transformations G is

$$\bar{x^i} = f^i(x, q, a), \quad f^i|_{a=0} = x^i,$$

$$\bar{q^\alpha} = \phi^\alpha(x, q, a), \quad \phi^\alpha|_{a=0} = q^\alpha. \tag{1.6}$$

According to the Lie's theory, the construction of the symmetry group G is equivalent to the determination of the corresponding *infinitesimal transformations* :

$$\bar{x^i} \approx x^i + a\,\xi^i(x,q), \quad \bar{q^\alpha} \approx q^\alpha + a\,\eta^\alpha(x,q) \tag{1.7}$$

obtained from (1.1) by expanding the functions f^i and ϕ^α into Taylor series in a about $a = 0$ and also taking into account the initial conditions

$$f^i\big|_{a=0} = x^i, \quad \phi^\alpha\big|_{a=0} = q^\alpha.$$

Thus, we have

$$\xi^i(x,q) = \left.\frac{\partial f^i}{\partial a}\right|_{a=0}, \quad \eta^\alpha(x,q) = \left.\frac{\partial \phi^\alpha}{\partial a}\right|_{a=0}. \tag{1.8}$$

One can now introduce the *symbol* of the infinitesimal transformations by writing (1.7) as

$$\bar{x^i} \approx (1 + a\,X)x, \quad \bar{q^\alpha} \approx (1 + a\,X)q,$$

where

$$X = \xi^i(x,q)\frac{\partial}{\partial x^i} + \eta^\alpha(x,q)\frac{\partial}{\partial q^\alpha}. \tag{1.9}$$

This differential operator X is known as the infinitesimal operator or generator of the group G. If the group G is admitted by (1.5), we say that X is an *admitted operator* of (1.5) or X is an *infinitesimal symmetry* of equation (1.5).

We now see how the derivatives are transformed.

The D_i transforms as

$$D_i = D_i(f^j)\bar{D}_j, \tag{1.10}$$

where $\bar{D}_j$ is the total differentiations in transformed variables $\bar{x}^i$. So

$$\bar{q}^\alpha_i = \bar{D}_j(q^\alpha), \qquad \bar{q}^\alpha_{ij} = \bar{D}_j(\bar{q}^\alpha_i) = \bar{D}_i(\bar{q}^\alpha_j), \cdots.$$

Now let us apply (1.10) and (1.6)

$$\begin{aligned} D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{q}^\alpha) \\ &= D_i(f^j)\bar{q}^\alpha_j. \end{aligned} \tag{1.11}$$

This

$$\left(\frac{\partial f^j}{\partial x^i} + q_i^\beta \frac{\partial f^j}{\partial q^\beta}\right)\bar{q}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + q_i^\beta \frac{\partial \phi^\alpha}{\partial q^\beta}. \tag{1.12}$$

The quantities $\bar{q}_j^\alpha$ can be represented as functions of $x, q, q_{(i)}, a$ for small a, ie., (1.12) is locally invertible:

$$\bar{q}_i^\alpha = \psi_i^\alpha(x, q, q_{(1)}, a), \quad \psi^\alpha|_{a=0} = q_i^\alpha. \tag{1.13}$$

The transformations in $x, q, q_{(1)}$ space given by (1.6) and (1.13) form a one-parameter group (one can prove this but we do not consider the proof) called the first prolongation or just extension of the group G and denoted by $G^{[1]}$.

We let

$$\bar{q}_i^\alpha \approx q_i^\alpha + a\zeta_i^\alpha \tag{1.14}$$

be the infinitesimal transformation of the first derivatives so that the infinitesimal transformation of the group $G^{[1]}$ is (3.12) and (1.14).

Higher-order prolongations of G, viz. $G^{[2]}$, $G^{[3]}$ can be obtained by derivatives of (1.11).

Prolonged generators

Using (1.11) together with (3.12) and (1.14) we get

$$\begin{aligned}
D_i(f^j)(\bar{q}_j^\alpha) &= D_i(\phi^\alpha) \\
D_i(x^j + a\xi^j)(q_j^\alpha + a\zeta_j^\alpha) &= D_i(q^\alpha + a\eta^\alpha) \\
(\delta_i^j + aD_i\xi^j)(q_j^\alpha + a\zeta_j^\alpha) &= q_i^\alpha + aD_i\eta^\alpha \\
q_i^\alpha + a\zeta_i^\alpha + aq_j^\alpha D_i\xi^j &= q_i^\alpha + aD_i\eta^\alpha \\
\zeta_i^\alpha &= D_i(\eta^\alpha) - q_j^\alpha D_i(\xi^j). \quad \text{(sum on } j\text{)}.
\end{aligned} \tag{1.15}$$

This is called the first prolongation formula. Likewise, one can obtain the second prolongation, viz.,

$$\zeta_{ij}^\alpha = D_j(\eta_i^\alpha) - q_{ik}^\alpha D_j(\xi^k), \quad \text{(sum on } k\text{)}. \tag{1.16}$$

By induction (recursively)

$$\zeta^{\alpha}_{i_1,i_2,\ldots,i_p} = D_{i_p}(\zeta^{\alpha}_{i_1,i_2,\ldots,i_{p-1}}) - q^{\alpha}_{i_1,i_2,\ldots,i_{p-1}j} D_{i_p}(\xi^j), \quad \text{(sum on } j). \tag{1.17}$$

The first and higher prolongations of the group G form a group denoted by $G^{[1]}, \cdots, G^{[p]}$. The corresponding prolonged generators are

$$X^{[1]} = X + \zeta^{\alpha}_i \frac{\partial}{\partial q^{\alpha}_i} \quad \text{(sum on } i, \alpha),$$

$$\vdots$$

$$X^{[p]} = X^{[p-1]} + \zeta^{\alpha}_{i_1,\ldots,i_p} \frac{\partial}{\partial q^{\alpha}_{i_1,\ldots,i_p}} \quad p \geq 1,$$

where

$$X = \xi^i(x,q)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,q)\frac{\partial}{\partial q^{\alpha}}.$$

1.4 Group admitted by a PDE

Definition 1.2 The vector field

$$X = \xi^i(x,q)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,q)\frac{\partial}{\partial q^{\alpha}}, \tag{1.18}$$

is a *point symmetry* of the pth-order PDE (1.5), if

$$X^{[p]}(E) = 0 \tag{1.19}$$

whenever $E = 0$. This can also be written as

$$X^{[p]} E\big|_{E=0} = 0, \tag{1.20}$$

where the symbol $|_{E=0}$ means evaluated on the equation $E = 0$.

Definition 1.3 Equation (1.19) is called the *determining equation* of (1.5) because it determines all the infinitesimal symmetries of (1.5).

Definition 1.4 (Symmetry group) A one-parameter group G of transformations (1.1) is called a symmetry group of equation (1.5) if (1.5) is form-invariant (has the same form) in the new variables $\bar{x}$ and $\bar{q}$, i.e.,

$$E(\bar{x}, \bar{q}, \bar{q_{(1)}}, \cdots, \bar{q_{(p)}}) = 0, \tag{1.21}$$

where the function E is the same as in equation (1.5).

1.5 Group invariants

Definition 1.5 A function $F(x, q)$ is called an *invariant of the group of transformation* (1.1) if

$$F(\bar{x}, \bar{q}) \equiv F(f^i(x, q, a), \phi^\alpha(x, q, a)) = F(x, q), \tag{1.22}$$

identically in x, q and a.

Theorem 1.2 (Infinitesimal criterion of invariance) A necessary and sufficient condition for a function $F(x, q)$ to be an invariant is that

$$X\,F \equiv \xi^i(x, q)\frac{\partial F}{\partial x^i} + \eta^\alpha(x, q)\frac{\partial F}{\partial q^\alpha} = 0\,. \tag{1.23}$$

It follows from the above theorem that every one-parameter group of point transformations (1.1) has n functionally independent invariants, which can be taken to be the left-hand side of any first integrals

$$J_1(x, q) = c_1, \cdots, J_n(x, q) = c_n$$

of the characteristic equations

$$\frac{dx^1}{\xi^1(x, q)} = \cdots = \frac{dx^n}{\xi^n(x, q)} = \frac{dq^1}{\eta^1(x, q)} = \cdots = \frac{dq^n}{\eta^n(x, q)}.$$

Theorem 1.3 If the infinitesimal transformation (3.12) or its symbol X is given, then the corresponding one-parameter group G is obtained by solving the Lie equations

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{q}), \quad \frac{d\bar{q}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{q}) \tag{1.24}$$

subject to the initial conditions

$$\bar{x}^i\big|_{a=0} = x, \quad \bar{q}^\alpha\big|_{a=0} = q\,.$$

1.6 Lie algebra

Let us consider two operators X_1 and X_2 defined by

$$X_1 = \xi_1^i(x, q)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x, q)\frac{\partial}{\partial q^\alpha}$$

and

$$X_2 = \xi_2^i(x, q)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x, q)\frac{\partial}{\partial q^\alpha}.$$

Definition 1.6 The *commutator* of X_1 and X_2, written as $[X_1, X_2]$, is defined by $[X_1, X_2] = X_1(X_2) - X_2(X_1)$.

Definition 1.7 A Lie algebra is a vector space L (over the field of real numbers) of operators $X = \xi^i(x, q)\frac{\partial}{\partial x^i} + \eta^\alpha(x, q)\frac{\partial}{\partial q}$ with the following property. If the operators

$$X_1 = \xi_1^i(x, q)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x, q)\frac{\partial}{\partial q}, \quad X_2 = \xi_2^i(x, q)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x, q)\frac{\partial}{\partial q}$$

are any elements of L, then their commutator

$$[X_1, X_2] = X_1(X_2) - X_2(X_1)$$

is also an element of L. It follows that the commutator is

1. Bilinear: for any $X, Y, Z \in L$ and $a, b \in \mathbb{R}$,

   $$[aX + bY, Z] = a[X, Z] + b[Y, Z], [X, aY + bZ] = a[X, Y] + b[X, Z];$$

2. Skew-symmetric: for any $X, Y \in L$,

$$[X, Y] = -[Y, X];$$

3. and satisfies the Jacobi identity: for any $X, Y, Z \in L$,

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0.$$

1.7 Conclusion

In this chapter we presented briefly some basic definitions and results of the Lie group analysis of PDEs. These included the algorithm to determine the Lie point symmetries of PDEs.

Chapter 2

Solutions of the ZK equation with power law nonlinearity in (3+1) dimensions

In this chapter, we first use the Lie symmetry analysis to find the group-invariant solutions of the ZK equation with power law nonlinearity in (3+1) dimensions given by

$$q_t + aq^n q_x + b\left(q_{xx} + q_{yy} + q_{zz}\right)_x = 0, \tag{2.1}$$

where a, b and n are constants. Here the first term represents the evolution term while a represents the coefficients of power law nonlinearity, and b is the coefficient of dispersion terms. The parameter n is the power law parameter while q is the wave profile. The independent variables x, y, z and t represent spatial and temporal variables respectively.

Subsequently, the extended tanh function method and the G'/G method are used to integrate the ZK equation. The soliton solution is obtained by the aid of ansatz method. There are numerical simulation to support the analytical development. This work is new and has been submitted for publication. See [29].

2.1 Symmetry analysis

In this section Lie point symmetries of the equation (2.1) are first calculated and then used to construct exact solutions.

2.1.1 Lie point symmetries

A Lie point symmetry of a differential equation is an invertible transformation of the dependent and independent variables that leaves the equation unchanged. Determining all the symmetries of a differential equation is a formidable task. However, Sophus Lie (1842-1899) realized that if we restrict ourself to symmetries that depend continuously on a small parameter and that form a group (continuous one-parameter group of transformations), one can linearize the symmetry conditions and end up with an algorithm for calculating continuous symmetries.

The symmetry group of the ZK equation (2.1), viz.,

$$q_t + aq^n q_x + b\left(q_{xx} + q_{yy} + q_{zz}\right)_x = 0.$$

will be generated by vector field of the form

$$\Gamma = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial z} + \xi^4\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial q},$$

where ξ^i, $i = 1, 2, 3, 4$ and η depend on x, y, z, t and q. Applying the third prolongation $\mathrm{pr}^{(3)}\Gamma$ to (2.1) and then solving the resultant overdetermined system of linear partial differential equations (PDEs) yields the following Lie point symmetries

$$\Gamma_1 = \frac{\partial}{\partial x}, \quad \Gamma_2 = \frac{\partial}{\partial y}, \quad \Gamma_3 = \frac{\partial}{\partial z}, \quad \Gamma_4 = \frac{\partial}{\partial t}, \quad \Gamma_5 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$

and

$$\Gamma_6 = nx\frac{\partial}{\partial x} + ny\frac{\partial}{\partial y} + nz\frac{\partial}{\partial z} + 3nt\frac{\partial}{\partial t} - 2q\frac{\partial}{\partial q}.$$

The commutation relations between these vector fields is given by the following table, the entry in row i and column j representing $[\Gamma_i, \Gamma_j]$:

Table 2.1: Commutator Table

$[\Gamma_i, \Gamma_j]$	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6
Γ_1	0	0	0	0	$-\Gamma_2$	Γ_1
Γ_2	0	0	0	0	Γ_1	Γ_2
Γ_3	0	0	0	0	0	Γ_3
Γ_4	0	0	0	0	0	$3\Gamma_4$
Γ_5	Γ_2	$-\Gamma_1$	0	0	0	0
Γ_6	$-\Gamma_1$	$-\Gamma_2$	$-\Gamma_3$	$-3\Gamma_4$	0	0

2.1.2 Exact solutions

One of the main purposes for calculating symmetries of a differential equation is to use them for obtaining symmetry reductions and finding exact solutions. In this subsection we will use the symmetries calculated in the previous subsection to obtain exact solutions of the ZK equation (2.1).

One way to derive exact solutions of (2.1) is by reducing it to an ordinary differential equation (ODE). This can be achieved with the use of Lie point symmetries admitted by (2.1). It is well known that the reduction of a partial differential equation with respect to r-dimensional (solvable) subalgebra of its Lie symmetry algebra leads to reducing the number of independent variables by r.

We now consider the symmetry $\Gamma_1 + \Gamma_2 + \Gamma_3$ and reduce the ZK equation (2.1) to a PDE in three independent variables. This symmetry yields the following four invariants:

$$f = z - y, \quad g = t, \quad h = x - y, \quad \theta = q.$$

Treating θ as the new dependent variable and f, g and h as new independent variables, the ZK equation (2.1) transforms to

$$\theta_g + a\theta^n\theta_h + 2b\theta_{hhh} + 2b\theta_{ffh} + 2b\theta_{fhh} = 0, \tag{2.2}$$

which is a nonlinear partial differential equation (NLPDE) in three independent variables. We now further reduce (2.2) using its symmetries. It can be shown that equation (2.2) has the following four Lie point symmetries:

$$\begin{aligned}
\Upsilon_1 &= \frac{\partial}{\partial f}, \\
\Upsilon_2 &= \frac{\partial}{\partial g}, \\
\Upsilon_3 &= \frac{\partial}{\partial h}, \\
\Upsilon_4 &= nf\frac{\partial}{\partial f} + 3ng\frac{\partial}{\partial h} + nh\frac{\partial}{\partial h} - 2\theta\frac{\partial}{\partial \theta}.
\end{aligned}$$

The symmetry $\Upsilon_2 + \rho\Upsilon_3$ (ρ is a constant) yields the three invariants

$$r = f, \quad s = g - \rho h, \quad \phi = \theta,$$

which gives a group invariant solution $\phi = \phi(r, s)$ that satisfies a NLPDE in two independent variables, namely

$$\phi_s - a\rho\phi^n\phi_s - 2b\rho^3\phi_{sss} - 2b\rho\phi_{rrs} + 2b\rho^2\phi_{rss} = 0. \tag{2.3}$$

The symmetry algebra of (2.3) is generated by the vector fields

$$\Sigma_1 = \frac{\partial}{\partial r} \quad \text{and} \quad \Sigma_2 = \frac{\partial}{\partial s}.$$

The combination $\alpha\Sigma_1 + \Sigma_2$ (α is a constant) of the two symmetries Σ_1 and Σ_2 yields the following invariants

$$u = r - \alpha s, \quad \psi_1 = \phi$$

and consequently using these invariants (2.3) is transformed to the nonlinear third order ODE

$$[2b\rho^3\alpha^2 + 2b\rho^2\alpha + 2b\rho]\psi'''(u) + a\rho(\psi(u))^n\psi'(u) - \psi'(u) = 0 \tag{2.4}$$

which can be written as

$$B_1\psi'''(u) + B_2\psi^n(u)\psi'(u) + B_3\psi'(u) = 0, \tag{2.5}$$

where

$$B_1 = 2b\rho^3\alpha^2 + 2b\rho^2\alpha + 2b\rho, \quad B_2 = a\rho, \quad B_3 = -1. \tag{2.6}$$

Integrating equation (2.5) twice with respect to u and taking the constants to be zero we obtain

$$\frac{B_1}{2}\psi'^2(u) + \frac{B_2}{(n+2)(n+1)}\psi^{n+2}(u) + \frac{B_3}{2}\psi^2(u) = 0. \tag{2.7}$$

This is a first-order variables separable equation. Integrating this equation and taking the constant of integration to be zero and reverting back to the original variables, we obtain the solution of the ZK equation (2.1) for arbitrary values of n in the form

$$q(x,y,z,t) = \left[\frac{(n+1)(n+2)}{2a\rho}\right]^{1/n} \operatorname{sech}^{2/n}(A_1), \tag{2.8}$$

where

$$A_1 = \frac{n\{\alpha\rho x - (\alpha\rho+1)y + z - \alpha t\}}{2\sqrt{2b\rho(\alpha^2\rho^2+\alpha\rho+1)}}.$$

We now give profiles of the solution (2.8) for two specific values of n; namely $n=1$ and $n=2$.

By choosing $a=1, b=1, \rho=1, n=1, \alpha=1, y=0, z=0$, we have the following profile of solution (2.8).

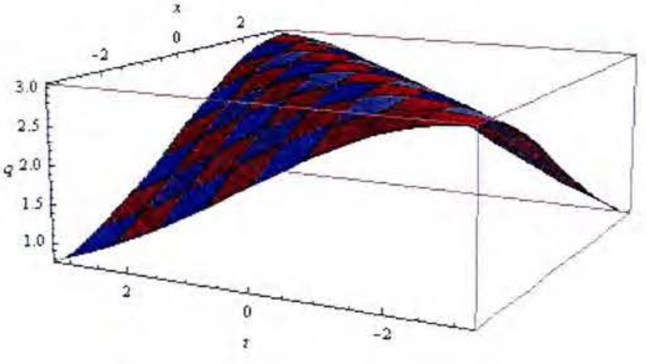

Figure 2.1: Profile of solution (2.8)

By choosing $a=1$, $b=1$, $\rho=1$, $n=2$, $\alpha=1$, $y=0$, $z=0$, we have the following profile of solution (2.8).

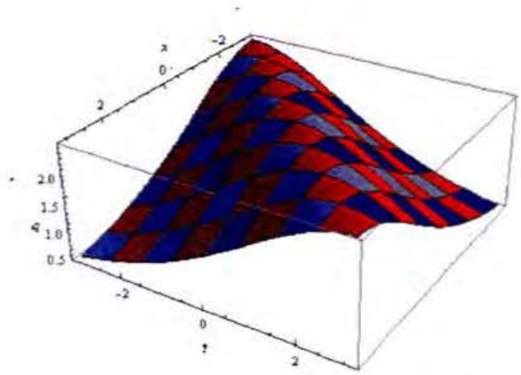

Figure 2.2: Profile of solution (2.8)

We now obtain group-invariant solutions of (2.1) for special cases, $n = 1$ and $n = 2$. By choosing $n = 1$ in (2.4) and solving the corresponding equation yields the following group invariant solutions of the ZK equation (2.1) for $n = 1$:

$$\begin{aligned} q(x, y, z, t) &= \frac{1}{a}\Bigg[16b\beta^2\left(\alpha^2\rho^2 + \alpha\rho + 1\right) - \frac{1}{\rho} \\ &\quad +24b\beta^2\left\{\alpha^2\rho^2 + \alpha\rho + 1\right\} \\ &\quad \left\{\cot^2\left(\beta u + \delta\right) + \tan^2\left(\beta u + \delta\right)\right\}\Bigg], \end{aligned} \tag{2.9}$$

$$\begin{aligned} q(x, y, z, t) &= \frac{1}{a}\Bigg[16b\beta^2\left(\alpha^2\rho^2 + \alpha\rho + 1\right) + \frac{1}{\rho} \\ &\quad -24b\beta^2\left\{\alpha^2\rho^2 + \alpha\rho + 1\right\} \\ &\quad \left\{\coth^2\left(\beta u + \delta\right) + \tanh^2\left(\beta u + \delta\right)\right\}\Bigg], \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} q(x, y, z, t) &= \frac{1}{a}\Bigg[24b\beta^2\omega^2\left(\alpha^2\rho^2 + \alpha\rho + 1\right) cn^2(\beta u|\omega) \\ &\quad + \frac{24b\beta^2\left(\omega^2 - 1\right)\left(\alpha^2\rho^2 + \alpha\rho + 1\right)}{cn^2(\beta u|\omega)} \\ &\quad - \frac{8b\beta^2\rho\left(2\omega^2 - 1\right)\left(\alpha^2\rho^2 + \alpha\rho + 1\right) - 1}{\rho}\Bigg], \end{aligned} \tag{2.11}$$

where $u = \alpha\rho x - (\alpha\rho + 1)y + z - \alpha t$ and α, β, δ are arbitrary constants. $\mathrm{cn}(Z|m)$ is the Jacobian elliptic function [30], which is defined as follows: If

$$Z = \int_0^\phi \frac{d\theta}{\sqrt{1 - m\sin^2\theta}},$$

where the angle ϕ is called the amplitude, then the function $\mathrm{cn}(Z|m)$ is defined as $\mathrm{cn}(Z|m) = \cos\phi$. Here m is called the modulus of the elliptic function and $0 \leq m \leq 1$.

By taking $n = 2$ in (2.4) and solving the corresponding equation we can obtain the following group invariant solutions of the ZK equation (2.1) for $n = 2$.

Note that $u = \alpha\rho x - (\alpha\rho + 1)y + z - \alpha t$ and δ is a constant in each of the following solutions.

$$q(x, y, z, t) = \frac{3}{4aA_1\rho}\cot(A_2 u + \delta) + A_1\tan(A_2 u + \delta), \tag{2.12}$$

where

$$A_1 = \sqrt{-\frac{3}{4a\rho}} \quad \text{and} \quad A_2 = \frac{1}{\sqrt{16b\rho\left(\alpha^2\rho^2 + \alpha\rho + 1\right)}}.$$

$$q(x, y, z, t) = \frac{\sqrt{3}}{2}\left[\frac{1}{a\rho A_3}\coth(A_4 u + \delta) + A_3\tanh(A_4 u + \delta)\right], \tag{2.13}$$

where

$$A_3 = \frac{1}{\sqrt{a\rho}} \quad \text{and} \quad A_4 = \sqrt{-\frac{1}{16(b\rho\left\{\alpha^2\rho^2 + \alpha\rho + 1\right\})}}.$$

$$q(x, y, z, t) = \sqrt{\frac{6\omega^2}{a\rho\left(\omega^2 + 1\right)}}\,sn\left[\sqrt{-\frac{1}{2b\rho\left(\omega^2 + 1\right)\left(\alpha^2\rho^2 + \alpha\rho + 1\right)}}\;u + \delta\,\middle|\,\omega\right] \tag{2.14}$$

where $\mathrm{sn}(Z|m) = \sin\phi$.

By taking $a = -1, b = 1, c = 1, \alpha = 1, \delta = 0, \omega = \frac{3}{4}, y = 0, z = 0$ we have the following plot of the solution (2.14).

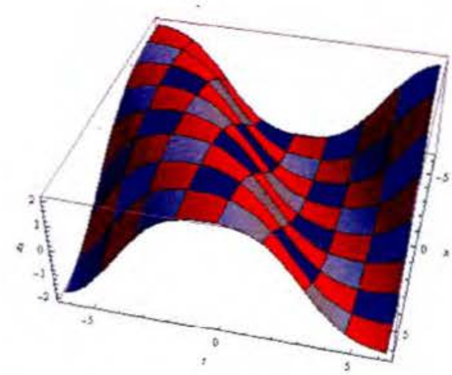

Figure 2.3: Profile of solution (2.14)

2.2 Extended Tanh-function method

In this section we use the extended tanh-function method [31] and obtain a few exact solutions of the ZK equation (2.1) for $n = 1$. We rewrite (2.4) for $n = 1$ as

$$B_0\psi'''(u) + B_1\psi'(u)\psi(u) + B_2\psi'(u) = 0. \tag{2.15}$$

where

$$B_0 = 2b\alpha^2\rho^3 + 2b\alpha\rho^2 + 2b\rho, \quad B_1 = a\rho, \quad B_2 = -1$$

with

$$u = \alpha\rho x - (\alpha\rho + 1)y + z - \alpha t.$$

By taking $n = 1$, let us consider the solution of (2.15) in the following form

$$\psi(u) = \sum_{i=0}^{M} A_i(G(u))^i, \tag{2.16}$$

where $(G(u))^i = \tanh^i(u)$, M is a positive integer that can be determined by balancing the highest order derivative with the highest nonlinear terms in equation and $A_0, \cdots, A_M$ are parameters to be determined. The crucial step of the method is to take full advantage of a Riccati equation that the tanh function satisfies and use its solutions to construct exact solutions.

The required Riccati equation is written as

$$G' = d + G^2. \tag{2.17}$$

The balancing procedure yields $M = 2$ so the solutions in (2.16) are of the form

$$\psi(u) = A_0 + A_1 G + A_2 G^2. \tag{2.18}$$

Substituting (2.17) and (2.18) into (2.15), we obtain algebraic system of equations in terms of A_0, A_1, A_2 by equating all coefficients of the functions G^i to zero. The corresponding algebraic equations are

$$\begin{aligned}
2A_1B_0d^2 + A_0A_1B_1d + A_1B_2d &= 0, \\
16A_2B_0d^2 + A_1^2B_1d + 2A_0A_2B_1d + 2A_2B_2d &= 0, \\
8A_1B_0d + 3A_1A_2B_1d + A_0A_1B_1 + A_1B_2 &= 0, \\
40A_2B_0d + 2A_2^2B_1d + A_1^2B_1 + 2A_0A_2B_1 + 2A_2B_2 &= 0, \\
6A_1B_0 + 3A_1A_2B_1 &= 0, \\
2A_2^2B_1 + 24A_2B_0 &= 0.
\end{aligned}$$

Solving the system of algebraic equations with the aid of Mathematica, we have the following cases:

Case 1.

$$A_0 = \frac{-8B_0d - B_2}{B_1}, \quad A_1 = 0, \quad A_2 = -\frac{12B_0}{B_1}.$$

$$q(x,y,z,t) = \frac{1}{B_1}\left[12B_0d\tanh^2\left(\sqrt{-d}u\right) - 8B_0d - B_2\right], \tag{2.19}$$

$$q(x,y,z,t) = \frac{1}{B_1}\left[12B_0d\coth^2\left(\sqrt{-d}u\right) - 8B_0d - B_2\right], \tag{2.20}$$

$$q(x,y,z,t) = \frac{1}{B_1}\left[-12B_0d\tan^2\left(\sqrt{d}u\right) - 8B_0d - B_2\right], \tag{2.21}$$

$$q(x,y,z,t) = \frac{1}{B_1}\left[-12B_0d\cot^2\left(\sqrt{d}u\right) - 8B_0d - B_2\right]. \tag{2.22}$$

Case 2.

$$d = 0, \quad A_0 = -\frac{B_2}{B_1}, \quad A_1 = 0, \quad A_2 = -\frac{12B_0}{B_1},$$

$$q(x, y, z, t) = -\frac{1}{B_1 u^2}\left[B_2 u^2 + 12B_0\right]. \tag{2.23}$$

2.3 (G'/G) expansion method

In this section we use the (G'/G)-expansion method [32] to obtain a few exact solutions of the ZK equation (2.1) for $n = 1$.

By taking $n = 1$, let us consider the solution of (2.15) in the following form:

$$\psi(u) = \sum_{i=0}^{M} A_i \left(\frac{G'(u)}{G(u)}\right)^i, \tag{2.24}$$

where $G(u)$ satisfies

$$G'' + \lambda G' + \mu G = 0 \tag{2.25}$$

with λ and μ constants. The positive integer M will be determined by the homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (2.15) where $A_0, \cdots, A_M$ are parameters to be determined.

The balancing procedure yields $M = 2$, so the solutions in (2.15) are of the form

$$\psi(u) = A_0 + A_1\left(\frac{G'(u)}{G(u)}\right) + A_2\left(\frac{G'(u)}{G(u)}\right)^2. \tag{2.26}$$

Substituting (2.25) and (2.26) into (2.15), we obtain algebraic system of equations in terms of A_0, A_1, A_2 by equating all coefficients of the functions $(G'(u)/G(u))^i$ to zero. The corresponding algebraic equations are

$$A_1B_0\lambda^2(-\mu)-6A_2B_0\lambda\mu^2-2A_1B_0\mu^2-A_0A_1B_1\mu-A_1B_2\mu=0,$$

$$-A_1B_0\lambda^3-14A_2B_0\lambda^2\mu-8A_1B_0\lambda\mu-A_0A_1B_1\lambda-A_1B_2\lambda$$
$$-16A_2B_0\mu^2-A_1^2B_1\mu-2A_0A_2B_1\mu-2A_2B_2\mu=0,$$

$$-8A_2B_0\lambda^3-7A_1B_0\lambda^2-52A_2B_0\lambda\mu-A_1^2B_1\lambda-2A_0A_2B_1\lambda$$
$$-2A_2B_2\lambda-8A_1B_0\mu-3A_1A_2B_1\mu-A_0A_1B_1-A_1B_2=0,$$

$$-38A_2B_0\lambda^2-12A_1B_0\lambda-3A_1A_2B_1\lambda-40A_2B_0\mu$$
$$-2A_2^2B_1\mu-A_1^2B_1-2A_0A_2B_1-2A_2B_2=0,$$

$$-2A_2^2B_1\lambda-54A_2B_0\lambda-3A_1A_2B_1-6A_1B_0=0,$$

$$-2A_2^2B_1-24A_2B_0=0.$$

Solving the system of algebraic equations with the aid of Mathematica, we have the following :

$$A_0=\frac{-\lambda^2B_0-8\mu B_0-B_2}{B_1},\quad A_1=-\frac{12\lambda B_0}{B_1},\quad A_2=\frac{A_1}{\lambda}.$$

When $\lambda^2-4\mu>0$,

$$\begin{aligned} q(x,y,z,t) &= \frac{1}{\chi_1}\Big[C_1^2\left(B_0\left(\lambda^2-4\mu\right)+B_2\right)\left(-e^{2u\sqrt{\lambda^2-4\mu}}\right) \\ &\quad +2C_2C_1\left(5B_0\left(\lambda^2-4\mu\right)-B_2\right)e^{u\sqrt{\lambda^2-4\mu}} \\ &\quad -C_2^2\left(B_0\left(\lambda^2-4\mu\right)+B_2\right)\Big], \end{aligned} \tag{2.27}$$

where

$$\chi_1=B_1\left(C_1e^{u\sqrt{\lambda^2-4\mu}}+C_2\right)^2.$$

When $\lambda^2 - 4\mu < 0$,

$$q(x,y,z,t) = \frac{1}{B_1}\Bigg[\frac{3B_0\left(C_1^2+C_2^2\right)\left(\lambda^2-4\mu\right)}{\left(C_1\sin\left(\frac{1}{2}u\sqrt{4\mu-\lambda^2}\right)+C_2\cos\left(\frac{1}{2}u\sqrt{4\mu-\lambda^2}\right)\right)^2} + B_0\left(-\left(\lambda^2-4\mu\right)\right) - B_2\Bigg]. \tag{2.28}$$

When $\lambda^2 - 4\mu = 0$,

$$q(x,y,z,t) = -\frac{1}{B_1}\left[\frac{12B_0C_1^2}{\left(C_1u+C_2\right)^2} + B_2\right]. \tag{2.29}$$

2.4 Solitary wave ansatz method; Soliton solution

In this section we will focus on obtaining the 1-soliton solution of (2.1) by the aid of solitary wave ansatz method. It needs to be noted that this method has been employed to carry out the integration of many NLEEs [14, 15, 16].

The solitary wave ansatz for the 1-soliton solution of (2.1) is taken to be

$$q(x,y,z,t) = A\,\mathrm{sech}^p\tau, \tag{2.30}$$

where

$$\tau = B_1x + B_2y + B_3z - vt. \tag{2.31}$$

In (2.30) and (2.31), A represents the soliton amplitude, B_i for $i = 1, 2, 3$ represents the inverse width of the soliton, v is the soliton velocity and the exponential p, which is unknown at this point, will be determined. Thus, from (2.30), we obtain

$$
\begin{aligned}
q_t &= Avp \operatorname{sech}^p\tau \tanh\tau, && (2.32)\\
q_x &= AB_1p \operatorname{sech}^p\tau \tanh\tau, && (2.33)\\
q^n q_x &= apB_1A^{n+1} \operatorname{sech}^{p(n+1)}\tau, && (2.34)\\
q_{xxx} &= -p^3AB_1^3\operatorname{sech}^p\tau\tanh\tau \\
&\quad +p(p+1)(p+2)AB_1^3\operatorname{sech}^{p+2}\tau\tanh\tau, && (2.35)\\
q_{xyy} &= -p^3AB_1B_2^2\operatorname{sech}^p\tau\tanh\tau \\
&\quad -p(p+1)(p+2)AB_1B_2^2\operatorname{sech}^{p+2}\tau\tanh\tau, && (2.36)\\
q_{xzz} &= -p^3AB_1B_3^2\operatorname{sech}^p\tau\tanh\tau \\
&\quad -p(p+1)(p+2)AB_1B_3^2\operatorname{sech}^{p+2}\tau\tanh\tau. && (2.37)
\end{aligned}
$$

Substituting (2.32)-(2.37) into (2.1), yields

$$
\begin{aligned}
&vp \operatorname{sech}^p\tau - apB_1A^n \operatorname{sech}^{p(n+1)}\tau + b\Big[-p^3B_1\operatorname{sech}^p\tau(B_1^2+B_2^2+B_3^2) \\
&+p(p+1)(p+2)B_1\operatorname{sech}^{p+2}\tau(B_1^2-B_2^2-B_3^2)\Big] = 0. \qquad (2.38)
\end{aligned}
$$

Equating the exponents $p(n+1)$ and $p+2$, we have

$$p(n+1) = p+2, \qquad (2.39)$$

which leads to

$$p = \frac{2}{n}. \qquad (2.40)$$

From (2.38) setting the respective coefficients of the linearly independent functions $\operatorname{sech}^p\tau$ and $\operatorname{sech}^{p+2}\tau$ to zero yields

$$A = \left[\frac{2b(n+1)(n+2)(B_1^2+B_2^2+B_3^2)}{an^2}\right]^{1/n} \qquad (2.41)$$

and

$$v = \frac{4bB_1(B_1^2+B_2^2+B_3^2)}{n^2}. \qquad (2.42)$$

Thus, the 1-soliton solution to (2.1) is given by

$$q(x,y,z,t) = A\operatorname{sech}^{2/n}\tau, \qquad (2.43)$$

where the amplitude A is given by (2.41) and the velocity v by (2.42).

Figure 4 below shows the profile of a 1- soliton solution (3.22) with $n = 1$ and for $a = 1$, $b = 1$, $B_1 = 1$, $B_2 = 1$, $B_3 = 1$, $t = 0$, $z = 0$.

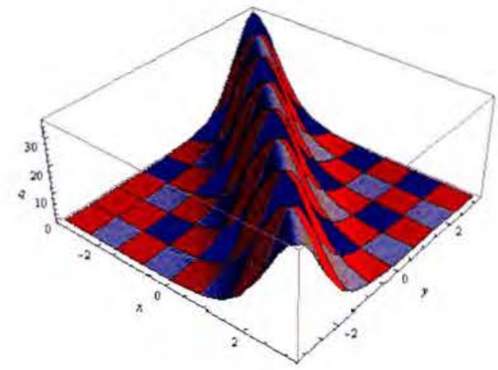

Figure 2.4: Profile of solution (3.22)

The profile of a 1- soliton solution (3.22) with $n = 2$ and for $a = 1$, $b = 1$, $B_1 = 1$, $B_2 = 1$, $B_3 = 1$, $t = 0$, $z = 0$ is given in Figure 5.

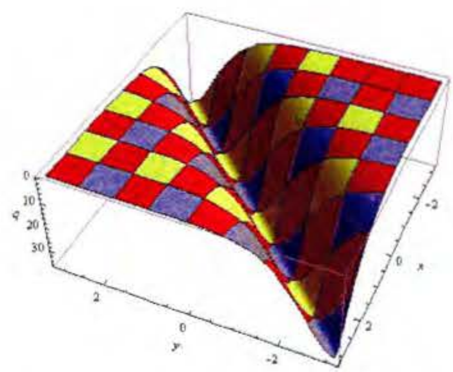

Figure 2.5: Profile of solution (3.22)

2.5 Conclusion

In this chapter we studied the Zakharov-Kuznestsov (ZK) equation with power law nonlinearity in (3+1) dimensions. Several solutions of this equation were obtained by employing various modern methods of integrability, i.e., Lie group method, (G'/G) method, extended tanh-function method and solitary wave ansatz method. The numerical simulations are also given to supplement the theory. The solutions obtained are cnoidal waves, periodic solutions, singular periodic solutions and solitary wave solutions.

Chapter 3

Solutions of a nonlinear flow problem

In this chapter a nonlinear flow problem of an incompressible viscous fluid is studied. The fluid is taken in a channel having two weakly permeable moving porous walls. An incompressible fluid fills the porous space inside the channel. The fluid is magnetohydrodynamic in the presence of a time-dependent magnetic field. Lie group method is applied in the derivation of analytic solution. The effects of the magnetic field, porous medium, permeation Reynolds number and wall dilation rate on the axial velocity are shown and discussed. The work of this Chapter has been accepted for publication. See [33].

3.1 Introduction

In many applications the two-dimensional flow of viscous fluid in a porous channel appears to be very useful. Many experimental and theoretical attempts have been made in the past. For example, Berman [19] studied the steady flow in a channel with stationary walls and small Reynolds number R_e. Majdalani et al. [21] considered the two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability. Their study focused on the viscous flow driven by small wall contractions and expansions of two weakly permeable walls. Based on double perturbations in the permeation Reynolds number Re and wall dilation rate α, they carried out their analytical procedure. Boutros et al. [22] studied the solution of the Navier-Stokes equations which described the unsteady incompressible laminar flow in a semi-infinite porous circular pipe with injection or suction through the pipe wall whose radius varies with time. The resulting fourth-order nonlinear differential equation was then solved using small-parameter perturbations. Asghar et al. [24] used the Lie group analysis to compute exact solution for the flow of viscous fluid through expanding-contracting channels.

The purpose of this research work is to generalize the flow analysis of [22] in two directions. The first generalization is concerned with the influence of variable magnetic field while the second accounts for the features of porous medium. Like in [22], the analytic solution for the arising nonlinear flow problem is studied by employing the Lie group method along with perturbation method, with R_e and α as the perturbation quantities. Finally, the graphs for self-axial velocity are plotted and discussed.

3.2 Problem statement

We consider an incompressible and magnetohydrodynamic (MHD) viscous fluid in a rectangular channel with walls of equal permeability. An incompressible fluid saturates the porous space between the two permeable walls which expand or contract uniformly at the rate α (the wall expansion ratio). In view of such configuration, symmetric nature of flow is taken into account at $y = 0$. Moreover, the fluid is electrically conducting in the presence of a variable magnetic field $(0, \delta H(t), 0)$. Here δ is the magnetic permeability and H is a magnetic field strength. The induced magnetic field is neglected under the assumption of small magnetic Reynolds number. The physical model of the flow is shown in Figure 1.

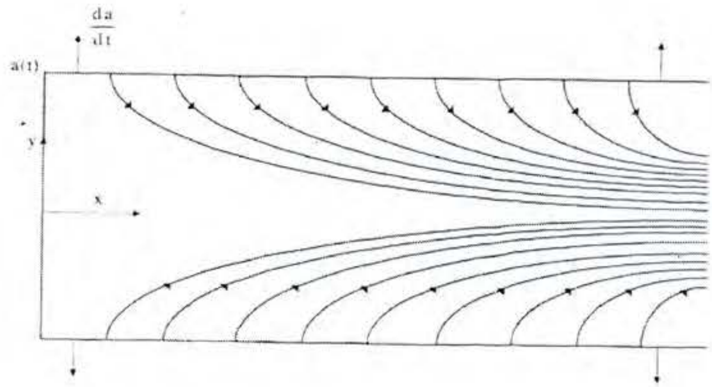

Figure 3.1: Coordinate system and bulk fluid motion

In view of the aforementioned assumptions, the governing equations can be written as

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{3.1}$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial \bar{x}} + s\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right]$$

$$-\frac{s\phi}{k}\bar{u} - \frac{r\delta^2 H^2(t)}{\rho}\bar{u}, \tag{3.2}$$

$$\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial \bar{y}} + s\left[\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right] - \frac{s\phi}{k}\bar{v}, \tag{3.3}$$

with the following conditions

$$\begin{aligned}
&\text{(i)} \quad && \bar{u} = 0, \quad \bar{v} = -V_w = -A\dot{a} \quad \text{at } \bar{y} = a(t), \\
&\text{(ii)} \quad && \frac{\partial \bar{u}}{\partial \bar{y}} = 0, \quad \bar{v} = 0 \quad \text{at } \bar{y} = 0, \\
&\text{(iii)} \quad && \bar{u} = 0 \quad \text{at } \bar{x} = 0.
\end{aligned} \tag{3.4}$$

In the above expressions $\bar{u}$ and $\bar{v}$ are the velocity components in $\bar{x}$ and $\bar{y}$-directions, respectively, ρ is the fluid density, $\bar{P}$ is the pressure, t is the time, s is the kinematic viscosity, ϕ and k are the porosity and permeability of porous medium, respectively, r is the electrical conductivity of fluid, V_w is the fluid inflow velocity, A is the injection coefficient corresponding to the porosity of wall and $\phi = V_f/V_c$ (where V_f and V_c, respectively, indicate the volume of the fluid and control volume).

The dimensional stream function $\overline{\Psi}(\bar{x}, \bar{y}, t)$ satisfies Eq.(3.1) according to the definitions of $\bar{u}$ and $\bar{v}$ given below

$$\bar{u} = \frac{\partial \overline{\Psi}}{\partial \bar{y}}, \qquad \bar{v} = -\frac{\partial \overline{\Psi}}{\partial \bar{x}},$$

which further takes the form

$$\bar{u} = \frac{1}{a}\frac{\partial \overline{\Psi}}{\partial y}, \qquad \bar{v} = -\frac{\partial \overline{\Psi}}{\partial \bar{x}}, \tag{3.5}$$

when $y = \bar{y}/a(t)$. Substituting Eq.(3.5) into Eqs.(3.2)-(3.4) and then relating the non-dimensional variables to the dimensional ones

$$\begin{aligned} &u = \frac{\bar{u}}{V_w}, \quad v = \frac{\bar{v}}{V_w}, \quad x = \frac{\bar{x}}{a(t)}, \quad \Psi = \frac{\overline{\Psi}}{aV_w}, \quad P = \frac{\bar{P}}{\rho V_w^2}, \\ &\bar{t} = \frac{tV_w}{a}, \quad \alpha = \frac{a\dot{a}}{s}, \quad N = \frac{r\delta^2 a}{\rho V_w}, \quad \frac{1}{R} = \frac{s\phi a}{kV_w}, \end{aligned} \tag{3.6}$$

we obtain

$$\begin{aligned} &\Psi_{y\bar{t}} + \Psi_y \Psi_{xy} - \Psi_x \Psi_{yy} + P_x - \frac{1}{R_e}[\alpha \Psi_y + \alpha y \Psi_{yy} + \Psi_{xxy} + \Psi_{yyy}] \\ &+ \frac{1}{R}\Psi_y + NH^2(t)\Psi_y = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\Psi_{x\bar{t}} + \Psi_y \Psi_{xx} - \Psi_x \Psi_{xy} - P_y - \frac{1}{R_e}[\alpha y \Psi_{xy} + \Psi_{xyy} + \Psi_{xxx}] \\ &+ \frac{1}{R}\Psi_x = 0 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\text{(i)} \quad \Psi_y = 0, \quad \Psi_x = 1 \quad \text{at } y = 1, \\ &\text{(ii)} \quad \Psi_{yy} = 0, \Psi_x = 0 \quad \text{at } y = 0, \\ &\text{(iii)} \quad \Psi_y = 0 \quad \text{at } x = 0, \end{aligned} \tag{3.9}$$

where

$$u = \Psi_y, \qquad v = -\Psi_x \tag{3.10}$$

and subscripts denote the partial derivatives, N is the magnetic parameter, $R_e(=aV_w/s)$ is the permeation Reynolds number and R is porosity parameter. It should be pointed out that the present problem reduces to the problem studied in [22] when $N=0$ and $R\rightarrow\infty$. Further $a\dot{a}=$ constant and $\alpha=a\dot{a}/s$, which implies that $a=(1+2.5\alpha t a_0^{-2})^{1/2}$. Here a_0 denotes the initial channel height.

3.3 Solution

In this section we solve the present problem by following closely the Lie group method in [22] under which equations (3.7) and (3.8) remain invariant. Following the methodology and notations in subsection (3.1) of [22] we note that the difference only occurs in the definitions of Δ_1 and Δ_2. In order to avoid repetition we only write the values of Δ_1 and Δ_2 here as

$$\begin{aligned}\Delta_1 &= \Psi_{y\bar{t}}+\Psi_y\Psi_{xy}-\Psi_x\Psi_{yy}+P_x-\frac{1}{R_e}[\alpha\Psi_y+\alpha y\Psi_{yy}+\Psi_{xxy}+\Psi_{yyy}]\\ &\quad+\frac{1}{R}\Psi_y+NH^2(t)\Psi_y,\\ \Delta_2 &= \Psi_{x\bar{t}}+\Psi_y\Psi_{xx}-\Psi_x\Psi_{xy}-P_y-\frac{1}{R_e}[\alpha y\Psi_{xy}+\Psi_{xyy}+\Psi_{xxx}]+\frac{1}{R}\Psi_x,\end{aligned}\tag{3.11}$$

where for other definitions and calculations, the readers may consult [22]. Now following the detailed procedure as given in [22] we finally obtain

$$\begin{aligned}&-K\frac{d^3h}{dy^3}+\Big[-\alpha Ky-hK_1-3KK_2\Big]\frac{d^2h}{dy^2}\\ &+\Big[-\alpha K-2\alpha KyK_2-hK_3+hK_4-KK_5-3KK_6+\frac{1}{R}+N\Big]\frac{dh}{dy}\\ &K_1\Big(\frac{dh}{dy}\Big)^2+\Big[-\alpha KK_2+\frac{1}{R}K_2+NK_2-\alpha KK_6y-KK_9-KK_{10}\Big]h\\ &+\Big[K_7-K_8\Big]h^2+\frac{1}{H}\frac{d\Gamma}{dx}=0,\end{aligned}\tag{3.12}$$

where

$$
\begin{aligned}
&K_1 = H_x, \quad K_2 = \frac{H_y}{H}, \quad K_3 = \frac{H_x H_y}{H}, \quad K_4 = H_{xy}, \\
&K_5 = \frac{H_{xx}}{H}, \quad K_6 = \frac{H_{yy}}{H}, \quad K_7 = \frac{H_y H_{xy}}{H}, \quad K_8 = \frac{H_x H_{yy}}{H}, \\
&K_9 = \frac{H_{xxy}}{H}, \quad K_{10} = \frac{H_{yyy}}{H},
\end{aligned} \tag{3.13}
$$

with

$$
u = x\frac{dG}{dy}, \quad v = -G \tag{3.14}
$$

and G satisfies

$$
\begin{aligned}
&\frac{d^4G}{dy^4} + \alpha\left[y\frac{d^3G}{dy^3} + 2\frac{d^2G}{dy^2}\right] + R_e G\frac{d^3G}{dy^3} - R_e/R\,\frac{d^2G}{dy^2} - R_e\frac{d^2G}{dy^2}N \\
&-R_e\frac{dG}{dy}\frac{d^2G}{dy^2} = 0
\end{aligned} \tag{3.15}
$$

along with

$$
\text{(i) } \frac{dG(1)}{dy} = 0, \quad \text{(ii) } G(1) = 1, \quad \text{(iii) } \frac{d^2G(0)}{dy^2} = 0, \quad \text{(iv) } G(0) = 0 \tag{3.16}
$$

and $K = R_e$. Writing

$$
\begin{aligned}
G &= G_1 + R_e G_2 + R_e^2 G_3 + 0(R_e^3), \\
G_1 &= G_{10} + \alpha G_{11} + \alpha^2 G_{12} + 0(\alpha^3), \\
G_2 &= G_{20} + \alpha G_{21} + \alpha^2 G_{22} + 0(\alpha^3), \\
G_3 &= G_{30} + \alpha G_{31} + \alpha^2 G_{32} + 0(\alpha^3),
\end{aligned}
$$

we solve the problem consisting of equation (3.15) and conditions given in (3.16) using second-order double perturbation and finally arrive at

$$
G_1(y) = \frac{1}{2800}\left[y(-(25y^2-13)(y^2-1)^2\alpha^2 + 210(y^2-1)^2\alpha - 1400(y^2-3))\right], \tag{3.17}
$$

$$
\begin{aligned}
G_2(y) = \frac{1}{232848000R}\Big[&y(y^2-1)^2(831600(R(-7\mathrm{N}+y^2+2)-7) \\
&-2310\alpha(-2y^2((240\mathrm{N}-227)R+240)+(552\mathrm{N}+681)R+65Ry^4+552) \\
&+\alpha^2(-35y^4((3905\mathrm{N}-6561)R+3905)+2y^2((133595\mathrm{N}+50481)R+133595) \\
&-3((29953\mathrm{N}+114111)R+29953)+12600Ry^6))\Big]
\end{aligned} \tag{3.18}
$$

and

$$\begin{aligned}
G_3(y) &= \frac{y(y^2-1)^2}{127135008000R^2}\Big[1260\alpha(R^2(1001\mathrm{N}^2(5y^2-9)(25y^2-37)\\
&-26\mathrm{N}(875y^6+18305y^4+293y^2-51137)\\
&-4060y^8+63133y^6+357696y^4+427177y^2+394166)\\
&+26R(77\mathrm{N}(5y^2-9)(25y^2-37)-875y^6-18305y^4-293y^2+51137)\\
&+1001(5y^2-9)(25y^2-37))+\alpha^2(105Ry^8((6510\mathrm{N}-46873)R+6510)\\
&-42y^6(R(350\mathrm{N}((1339\mathrm{N}-7698)R+2678)+3099111R-2694300)+468650)\\
&+14y^4(R(900\mathrm{N}((6552\mathrm{N}-10585)R+13104)-2957491R-9526500)+5896800)\\
&-y^2(R(84\mathrm{N}((1262105\mathrm{N}+3260532)R+2524210)-95806709R+273884688)\\
&+106016820)+3R(42\mathrm{N}((245908\mathrm{N}+2413431)R+491816)+100425529R\\
&+101364102)+783825R^2y^{10}+30984408)+491400(R(7y^4((55\mathrm{N}-102)R+55)\\
&-2y^2(77\mathrm{N}((10\mathrm{N}-23)R+20)+530R)+77\mathrm{N}((44\mathrm{N}+69)R+88)+28Ry^6\\
&-1406R+1771(2y^2+3))+308(11-5y^2))\Big]. \qquad (3.19)
\end{aligned}$$

It can be easily noted that for $N = 0$ and $R \to \infty$, $G(y)$ reduces to the result presented in [22], provided we use a first-order double perturbation. This shows confidence in the present calculations. The shear stress at the wall with $y = 1$ is

$$\tau_w = Kx\frac{d^2G(1)}{dy^2}. \qquad (3.20)$$

The velocity components through Eqs.(3.14) and (3.19) are given by

$$u = x\frac{dG}{dy}, \qquad (3.21)$$

$$v = -G. \qquad (3.22)$$

3.4 Results and discussion

In this section we study the effects of magnetic field N, porous medium R, on self-axial velocity both analytically and numerically and the results are plotted. The numerical solution is obtained by using the shooting method, coupled with Runge-Kutta scheme.

A. Self-axial velocity

Figures 3.2 and 3.3 demonstrates the behaviour of the self axial velocity u/x for magnetic parameter $N = 0.5$, porosity parameter $R = 0.5$, permeation Reynolds number $R_e = -1$ and 1, at $-1 \leq \alpha \leq 1$. Figure 3.2 shows the case of $R_e = -1$. When $\alpha > 0$, the flow towards the centre becomes greater, this leads to the axial-velocity to be greater near the centre. We noticed that this behaviour changes when $\alpha < 0$, that is, the flow towards the centre results in lower axial velocity near the centre and higher near the wall.

Figure 3.3 shows the case of $R_e = 1$. When $\alpha > 0$, the flow towards the wall becomes greater, the axial-velocity is lesser near the centre. When $\alpha < 0$ changes, the flow towards the wall results in lower axial velocity near the wall and higher near the centre.

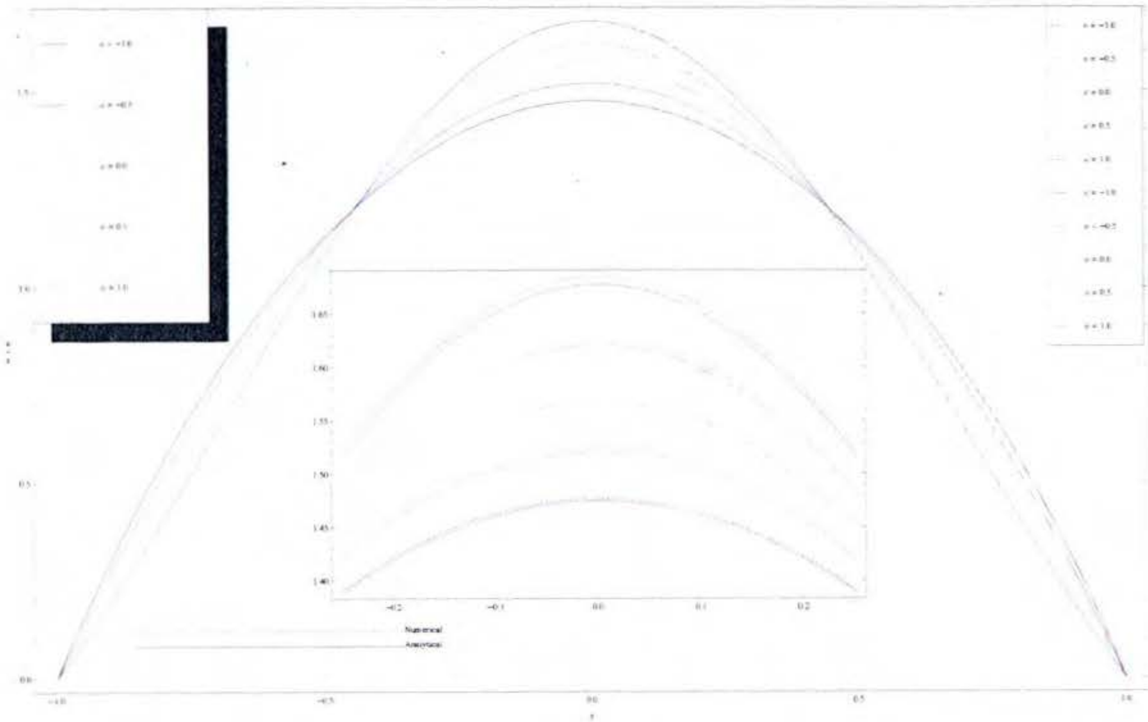

Figure 3.2: Self-axial velocity profiles over a range of α at $N = 0.5$, $R_e = -1$ and $R = 0.5$

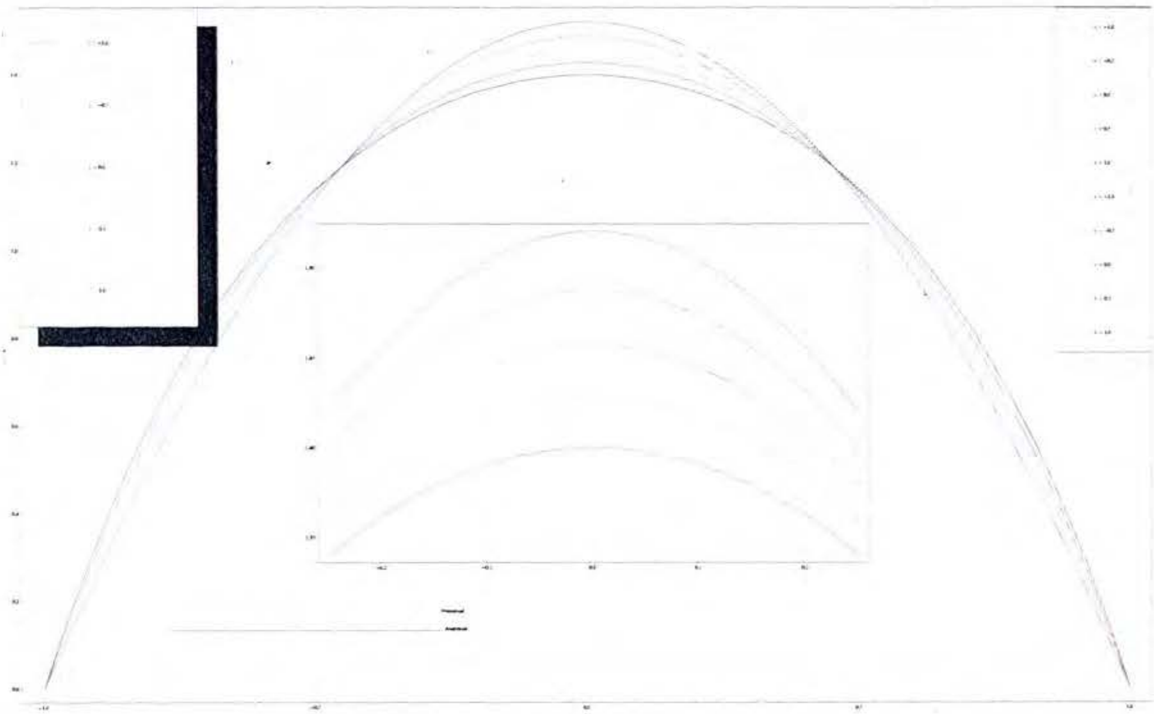

Figure 3.3: Self-axial velocity profiles over a range of α at $N = 0.5$, $R_e = 1$ and $R = 0.5$

From the figures above, we can see that the behaviour of the graphs is a cosine profile. Comparing analytical and numerical solutions, the percentage error increases as N increases for all $|\alpha|$, see Tables 3.1, 3.2 and 3.3.

Table 3.1: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.3$ for $R = 0.5, R_e = -1, \alpha = -0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$N = 0.5$	1.374237	1.375731	0.108609
$N = 1.0$	1.381895	1.384237	0.169198
$N = 1.5$	1.389799	1.393274	0.249420

Table 3.2: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.3$ for $R = 0.5, R_e = -1$ and $\alpha = 0.0$.

	Analytical Method	Numerical Method	Percentage Error (%)
$N = 0.5$	1.398273	1.400185	0.136611
$N = 1.0$	1.406663	1.409625	0.210186
$N = 1.5$	1.415323	1.419678	0.306770

Table 3.3: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.3$ for $R = 0.5, R_e = -1$ and $\alpha = 0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$N = 0.5$	1.423053	1.425483	0.170456
$N = 1.0$	1.432188	1.435905	0.258803
$N = 1.5$	1.441616	1.447026	0.373840

For porosity parameter R, the axial velocity and the percentage error between analytical and numerical solutions decreases as R increases, for the same $|\alpha|$, see Tables 3.4, 3.5 and 3.6.

Table 3.4: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.3$ for $N = 0.5, R_e = -1$ and $\alpha = -0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R = 0.5$	1.374237	1.375731	0.108609
$R = 1.0$	1.359664	1.360126	0.033979
$R = 1.5$	1.355025	1.355296	0.019936

Table 3.5: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.3$ for $N = 0.5, R_e = -1$ and $\alpha = 0.0$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R = 0.5$	1.398273	1.400185	0.136611
$R = 1.0$	1.382302	1.382914	0.044241
$R = 1.5$	1.377219	1.377581	0.026294

Table 3.6: Comparison between analytical and numerical solutions for self-axial velocity u/x at $y = 0.3$ for $N = 0.5, R_e = -1$ and $\alpha = 0.5$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R = 0.5$	1.423053	1.425483	0.170456
$R = 1.0$	1.405658	1.406468	0.057581
$R = 1.5$	1.400120	1.400610	0.035000

B. Shear stress

The figure below illustrate the effects of varying governing parameters on the character of the shear stress at the wall. For a suction-contracting process ($Re = -1$ and $\alpha < 0$), the shear stress is positive until expansion is sufficiently large, while for a suction-expansion process ($Re = 1$ and $\alpha > 0$) the shear stress turns negative.

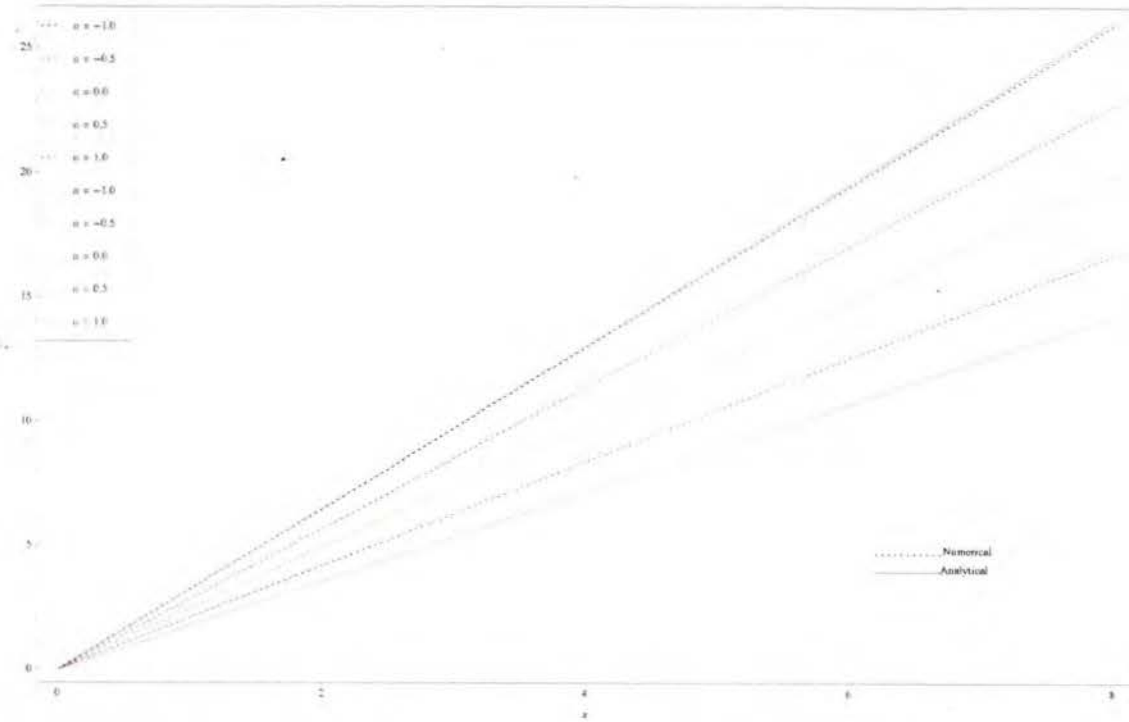

Figure 3.4: Shear stress profiles over a range of α at $N = 0.5$, $R_e = -1$ and $R = 0.5$

We noticed that, the wall shear stress decreases as the Reynolds number R_e increases, see Table 3.7.

Table 3.7: Comparison between analytical and numerical solutions for shear stress τ_ω at $x = 2$ for $N = 0.5$ and $\alpha = -1$.

	Analytical Method	Numerical Method	Percentage Error (%)
$R_e = -1$	6.526164	6.483047	0.665074
$R_e = 1$	-7.731125	-7.755944	0.320003

3.5 Conclusion

In this chapter, we have generalized the flow analysis of [22] with the influence of magnetic field and porous medium. The analytical solution for the arising nonlinear problem was obtained by using Lie symmetry technique in conjunction with a second-order double perturbation method. We have studied the effects of magnetic field (N) and porous medium (R) on the self-axial velocity and the results are plotted. We compared the analytical solution with the numerical solution for self-axial velocity at different values of N and R. We found that as N increases the self-axial velocity increases and as R increases the self-axial velocity decreases. Here we have noticed that the analytical results obtained matches quite well with the numerical results for a good range of these parameters. We also noticed that for all cases the self-axial velocity have the similar trend as in [22], that is, the axial velocity approaches a cosine profile. Finally, we observed that when $N = 0$ and R approaches infinity our problem reduces to the problem in [22] and our results (analytical and numerical) also reduce to the results in [22], with the use of first-order double perturbation method.

Chapter 4

Concluding remarks

In this research project Lie group method was applied to study two nonlinear partial differential equations arising in fluids.

In Chapter 1, a brief introduction to the Lie group theory of partial differential equations was given. This include the algorithm to determine the Lie point symmetries of partial differential equations.

Lie symmetry technique along with other methods of integrability, were used to carry out the integration of the ZK equation (2.1) with power law nonlinearity in (3+1) dimension in Chapter 2. Numerical simulations were also given to supplement the analytical development. This work was submitted for publication. See [29].

In Chapter 3, we generalized the flow analysis of [22] with the influence of magnetic field and porous medium. Lie symmetry analysis along with second-order double perturbation was applied to obtain the analytical solution. The effect of porous medium and magnetic field on axial velocity were shown and discussed. The work of this Chapter has appeared in [33].

In future we will use the Lie point symmetries of the ZK equation (2.1) obtained in this research project to construct conservation laws of (2.1).

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